

# GEOMETRY OF CARNOT–CARATHÉODORY SPACES, DIFFERENTIABILITY AND COAREA FORMULA<sup>1</sup>

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## Abstract

We give a simple proof of Gromov’s Theorem on nilpotentization of vector fields, and exhibit a new method for obtaining quantitative estimates of comparing geometries of two different local Carnot groups in Carnot–Carathéodory spaces with  $C^{1,\alpha}$ -smooth basis vector fields,  $\alpha \in [0, 1]$ . From here we obtain the similar estimates for comparing geometries of a Carnot–Carathéodory space and a local Carnot group. These two theorems imply basic results of the theory: Gromov type Local Approximation Theorems, and for  $\alpha > 0$  Rashevskii–Chow Theorem and Ball–Box Theorem, etc. We apply the obtained results for proving  $hc$ -differentiability of mappings of Carnot–Carathéodory spaces with continuous horizontal derivatives. The latter is used in proving the coarea formula for some classes of contact mappings of Carnot–Carathéodory spaces.

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# 1 Introduction

The geometry of Carnot–Carathéodory spaces naturally arises in the theory of subelliptic equations, contact geometry, optimal control theory, non-holonomic mechanics, neurobiology and other areas (see works by A. A. Agrachev [1], A. A. Agrachev and J.-P. Gauthier [3], A. A. Agrachev and A. Mari-go [4], A. A. Agrachev and A. V. Sarychev [5, 6, 7, 8, 9], A. Bellaïche [15], A. Bonfiglioli, E. Lanconelli and F. Uguzzoni [18], S. Buckley, P. Koskela and G. Lu [19], L. Capogna [21, 22], G. Citti, N. Garofalo and E. Lanconelli [31], L. Capogna, D. Danielli and N. Garofalo [23, 24, 25, 26, 27], Ya. Eliashberg [35, 36, 37, 38], G. B. Folland [44, 45], G. B. Folland and E. M. Stein [46], B. Franchi, R. Serapioni, F. Serra Cassano [55, 56, 57, 58], N. Garofalo [60], N. Garofalo and D.-M. Nhieu [62, 63], R. W. Goodman [65], M. Gromov [68, 69], L. Hörmander [74], F. Jean [75], V. Jurdjevic [82], G. P. Leonardi, S. Rigot [89], W. Liu and H. J. Sussman [91], G. Lu [92], G. A. Margulis and G. D. Mostow [99, 100], G. Metivier [101], J. Mitchell [102], R. Montgomery [103, 104], R. Monti [105, 106], A. Nagel,

F. Ricci, E. M. Stein [108, 109], A. Nagel, E. M. Stein and S. Wainger [110], P. Pansu [112, 113, 114, 115], L. P. Rothschild and E. M. Stein [119], R. S. Strichartz [122], A. M. Vershik and V. Ya. Gershkovich [124], S. K. Vodopyanov [125, 127, 128, 129, 130], S. K. Vodopyanov and A. V. Greshnov [131], C. J. Xu and C. Zuily [138] for an introduction to this theory and some its applications).

A Carnot–Carathéodory space (below referred to as a *Carnot manifolds*)  $\mathbb{M}$  (see, for example, [68, 124]) is a connected Riemannian manifold with a distinguished horizontal subbundle  $H\mathbb{M}$  in the tangent bundle  $T\mathbb{M}$  that meets some algebraic conditions on the commutators of vector fields  $\{X_1, \dots, X_n\}$  constituting a local basis in  $H\mathbb{M}$ ,  $n = \dim H\mathbb{M}$ .

The distance  $d_c$  (the intrinsic Carnot–Carathéodory metric) between points  $x, y \in \mathbb{M}$  is defined as the infimum of the lengths of *horizontal* curves joining  $x$  and  $y$  and is non-Riemannian if  $H\mathbb{M}$  is a proper subbundle (a piecewise smooth curve  $\gamma$  is called horizontal if  $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ ). See results on properties of this metric in the monograph by D. Yu. Burago, Yu. D. Burago and S. V. Ivanov [20].

The Carnot–Carathéodory metric is applied in the study of hypoelliptic operators, see C. Fefferman and D. H. Phong [43], L. Hörmander [74], D. Jerison [76], A. Nagel, E. M. Stein and S. Wainger [110], L. P. Rothschild and E. M. Stein [119], A. Sánchez-Calle [120]. Also, this metric and its properties are essentially used in theory of PDE’s (see papers by M. Biroli and U. Mosco [16, 17], S. M. Buckley, P. Koskela and G. Lu [19], L. Capogna, D. Danielli and N. Garofalo [23, 24, 25, 26, 27], V. M. Chernikov and S. K. Vodop’yanov [29], D. Danielli, N. Garofalo and D.-M. Nhieu [33], B. Franchi [47], B. Franchi, S. Gallot and R. Wheeden [48], B. Franchi, C. E. Gutiérrez and R. L. Wheeden [49], B. Franchi and E. Lanconelli [50, 51], B. Franchi, G. Lu and R. Wheeden [52, 53], B. Franchi and R. Serapioni [54], R. Garattini [59], N. Garofalo and E. Lanconelli [61], P. Hajłasz and P. Strzelecki [71], J. Jost [77, 78, 79, 80], J. Jost and C. J. Xu [81], S. Marchi [98], K. T. Sturm [123]).

The following results are usually regarded as foundations of the geometry of Carnot manifolds:

1. Rashevskii–Chow Theorem [30, 118] on connection of two points by a horizontal path;
2. Ball–Box Theorem [110] (saying that a ball in Carnot–Carathéodory metric contains a “box” and is a subset of a “box” with controlled “radii”);

3. Mitchell's Theorem [102] on convergence of rescaled Carnot–Carathéodory spaces around  $g \in \mathbb{M}$  to a nilpotent tangent cone;
4. Gromov's Theorem [68] on convergence of “rescaled” with respect to  $g \in \mathbb{M}$  basis vector fields to *nilpotentized* (at  $g$ ) vector fields generating a graded nilpotent Lie algebra (the corresponding connected and simply connected Lie group is called the *nilpotent tangent cone at  $g$* ); here  $g \in \mathbb{M}$  is an arbitrary point;
5. Gromov Approximation Theorem [68] on local comparison of Carnot–Carathéodory metrics in the initial space and in the nilpotent tangent cone, and its improvements due to A. Bellaïche [15].

The goal of the paper is both to give a new approach to the geometry of Carnot manifolds and to establish some basic results of geometric measure theory on these metric structures including an appropriate differentiability and a coarea formula.

New results in the geometry of Carnot manifolds contains essentially new quantitative estimates of closeness of geometries of different tangent cones located one near another. One of the peculiarities of the paper is that we solve all problems under minimal assumption on smoothness of the basis vector fields (they are  $C^{1,\alpha}$ -smooth,  $0 \leq \alpha \leq 1$ ), although all the basic results are new even for  $C^\infty$ -vector fields. In some parts of this paper, the symbol  $C^{1,\alpha}$  means that the derivatives of the basis vector fields are  $H^\alpha$ -continuous with respect to some nonnegative symmetric function  $\mathfrak{d} : U \times U \rightarrow \mathbb{R}$ ,  $U \subseteq \mathbb{M}$ , such that  $\mathfrak{d} \geq C\rho$ ,  $0 < C < \infty$ , where  $C$  depends only on  $U$ , and  $\rho$  is Riemannian distance. Some additional properties of  $\mathfrak{d}$  are described below when it is necessary. Note that from the very beginning it is unknown whether Rashevskii–Chow Theorem is true for  $C^{1,\alpha}$ -smooth basis vector fields. Therefore Carnot–Carathéodory distance can not be well defined. We use the quasimetric  $d_\infty$  instead of  $d_c$ , which is defined as follows: if  $y = \exp\left(\sum_{i=1}^N y_i X_i\right)(x)$ , then  $d_\infty(x, y) = \max_{i=1, \dots, N} \{|y_i|^{\frac{1}{\deg X_i}}\}$ , and in smooth case is equivalent to  $d_c$  [110, 68]. One of the main results is the following (see below Theorem 2.4.1 for sharp statement).

**Theorem 1.0.1.** *Suppose that  $d_\infty(u, u') = C\varepsilon$ ,  $d_\infty(u, v) = \mathcal{C}\varepsilon$  for some  $C, \mathcal{C} < \infty$ ,*

$$w_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v) \text{ and } w'_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\right)(v).$$

Then, for  $\alpha > 0$ , we have

$$\max\{d_\infty^u(w_\varepsilon, w'_\varepsilon), d_\infty^{u'}(w_\varepsilon, w'_\varepsilon)\} \leq L\varepsilon\rho(u, u')^{\frac{\alpha}{M}}$$

where  $L$  is uniformly bounded in  $u, u', v \in U \subset \mathbb{M}$ , and in  $\{w_i\}_{i=1}^N$  belonging to some compact neighborhood of 0.

In the case of  $\alpha = 0$ , we have

$$\max\{d_\infty^u(w_\varepsilon, w'_\varepsilon), d_\infty^{u'}(w_\varepsilon, w'_\varepsilon)\} \leq \varepsilon o(1)$$

where  $o(1)$  is uniform in  $u, u', v \in U \subset \mathbb{M}$ , and in  $\{w_i\}_{i=1}^N$  belonging to some compact neighborhood of 0 as  $\varepsilon \rightarrow 0$ .

Here we assume that  $U \subset \mathbb{M}$  is a compact neighborhood small enough and  $\rho$  is a Riemannian metric. The symbol  $\widehat{X}_i^u$  ( $\widehat{X}_i^{u'}$ ) denotes nilpotentized at  $u$  ( $u'$ ) vector fields (see item 4 above). These vector fields constitute Lie algebra of the nilpotent tangent cone at  $u$  ( $u'$ ).

Further, in Theorem 2.3.1 we extend this result to the case of a "chain" consisting of several points.

The content of obtained estimates is very profound: they imply both new properties of Carnot manifolds and the above-mentioned ones.

The investigation of sub-Riemannian geometry under minimal smoothness of the basis vector fields is motivated by the recently constructed by G. Citti and A. Sarti, and R. K. Hladky and S. D. Pauls model of visualization [32, 73]. More exactly, the model of a brain perception of a black-and-white plain image is constructed in these papers. This model makes possible the interpretation on a computer of a human brain's work during the visualization of information. In particular, it is shown how the human brain completes the image part of which is closed. The geometry of this model is based on a roto-translation group which is a three-dimensional Carnot manifold with a tangent cone being a Heisenberg group  $\mathbb{H}^1$  at each point. Since by now there are no theorems on regularity of the image created by a human brain, any reduction of smoothness of vector fields is essential for the construction of sharp visualization models.

The main result concerning the geometry of Carnot manifolds is proved in Section 2. The method of proving is new, and it essentially uses Hölder dependence of solutions to ordinary differential equations on parameter (see Theorem 2.1.13). Probably, this dependence is not a new result. For reader's convenience we give its independent proof in Section 5. In Subsection 2.1, all other auxiliary result are formulated.

In Subsection 2.2 we prove, in particular, the following statements

**A:** Let  $X_j \in C^1$  on a Carnot manifold  $\mathbb{M}$ . On  $\text{Box}(g, \varepsilon r_g)$ , consider the vector fields  $\{\varepsilon X_i\} = \{\varepsilon^{\deg X_i} X_i\}$ ,  $i = 1, \dots, N$ . Then the uniform convergence

$$X_i^\varepsilon = (\Delta_{\varepsilon^{-1}}^g)_*^\varepsilon X_i \rightarrow \widehat{X}_i^g \quad \text{as } \varepsilon \rightarrow 0, \quad i = 1, \dots, N,$$

holds at the points of the box  $\text{Box}(g, r_g)$  and this convergence is uniform in  $g$  belonging to some compact set, where the collection  $\{\widehat{X}_i^g\}$ ,  $i = 1, \dots, N$ , of vector fields around  $g$  constitutes a basis of a graded nilpotent Lie algebra;

**B:** There exists a constant  $Q = Q(U)$ ,  $U$  is a compact domain in  $\mathbb{M}$ , such that the inequality

$$d_\infty(u, v) \leq Q(d_\infty(u, w) + d_\infty(w, v))$$

holds for every triple of points  $u, w, v \in U$  where  $Q(U)$  depends on  $U$ .

**C:** Given points  $u, v \in \mathbb{M}$ ,  $d_\infty(u, v) = C\varepsilon$  for some  $C < \infty$ ,

$$w_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} X_i\right)(v) \quad \text{and} \quad w'_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v),$$

we have

$$\max\{d_\infty(w_\varepsilon, w'_\varepsilon), d_\infty^u(w_\varepsilon, w'_\varepsilon)\} \leq \varepsilon o(1)$$

where  $o(1)$  is uniform in  $u, v$  belonging to a compact neighborhood  $U \subset \mathbb{M}$ , and in  $\{w_i\}_{i=1}^N$  belonging to some compact neighborhood of 0 as  $\varepsilon \rightarrow 0$ .

Statement **A** is just Gromov's Theorem [68] on the nilpotentization of vector fields. Gromov has formulated it for  $C^1$ -smooth fields, however, Example 2.2.12 by Valeriĭ Berestovskii makes evident that arguments of the proof given in [68, pp. 128–133] have to be corrected. In Corollary 2.2.11 we give a new proof of this assertion based on another idea.

Statement **B** says that the quasimetric  $d_\infty$  meets the generalised triangle inequality. The implication **A**  $\implies$  **B** is proved in Corollary 2.2.14.

Statement **C** gives an estimate of divergence of integral lines of the given vector fields and the nilpotentized vector fields.

The implication **B**  $\implies$  **C** is a particular case of Theorem 2.7.1.

In theory developed in Subsection 2.4, is based on the generalized triangle inequality as a starting point.

In Subsection 2.4, we prove one of the basic results of Section 2, namely, Theorem 2.4.1 which compares local geometries of two different local Carnot groups. It is essentially based on the main theorem of Subsection 2.3 which compares "global" geometries of different tangent cones (i. e., it looks like Theorem 2.4.1 with  $\varepsilon = 1$ ). Subsection 2.5 is devoted to approximation

theorems. In particular, we compare metrics of two tangent cones, and the metric of a tangent cone with the initial one. There we give their proofs and the proofs of some auxiliary properties of the geometry. Further, in Subsection 2.6, we prove Theorem 2.3.1, which is the "continuation" of Theorem 2.4.1. In Subsection 2.7, we compare the geometry of a Carnot manifold with the one of a tangent cone. In Subsection 2.8, we give applications of our results to investigation of the sub-Riemannian geometry. We prove Gromov type theorem on the nilpotentization of vector fields [68], a new statement implying Rashevskii–Chow Theorem, Ball–Box Theorem, Mitchell Theorem on Hausdorff dimension of Carnot manifolds and many other corollaries.

Main results of Section 2 are formulated in short communications [132, 133].

Section 3 is devoted to differentiability of mappings in the category of Carnot manifolds.

We recall the classical definition of differentiability for a mapping  $f : \mathbb{M} \rightarrow \mathbb{N}$  of two Riemannian manifolds:  $f$  is differentiable at  $x \in \mathbb{M}$  if there exists a linear mapping  $L : T_x \mathbb{M} \rightarrow T_{f(x)} \mathbb{N}$  of the tangent spaces such that

$$\rho_{\mathbb{N}}(f(\exp_x h), \exp_{f(x)} Lh) = o(\|h\|_x), \quad h \in T_x \mathbb{M},$$

where  $\exp_x : T_x \mathbb{M} \rightarrow \mathbb{M}$  and  $\exp_{f(x)} : T_{f(x)} \mathbb{N} \rightarrow \mathbb{N}$  are the exponential mappings, and  $\rho_{\mathbb{N}}$  is the Riemannian metric in  $\mathbb{N}$ ,  $\|h\|_x$  is the length of  $h \in T_x \mathbb{M}$ .

It is known (see [68, 104]) that the local geometry of a Carnot manifold at a point  $g \in \mathbb{M}$  can be modelled as a graded nilpotent Lie group  $\mathbb{G}_g \mathbb{M}$ . It means that the tangent space  $T_g \mathbb{M}$  has an additional structure of a graded nilpotent Lie group. If  $\mathbb{M}$  and  $\mathbb{N}$  are two Carnot manifolds and  $f : \mathbb{M} \rightarrow \mathbb{N}$  is a mapping then a suitable concept of differentiability can be obtained from the previous concept in the following way:  $f$  is *hc*-differentiable at  $x \in \mathbb{M}$  if there exists a horizontal homomorphism  $L : \mathbb{G}_x \mathbb{M} \rightarrow \mathbb{G}_{f(x)} \mathbb{N}$  of the nilpotent tangent cones such that

$$\tilde{d}_c(f(\exp_x h), \exp_{f(x)} Lh) = o(|h|_x), \quad h \in \mathbb{G}_x \mathbb{M},$$

where  $\tilde{d}_c$  is the Carnot–Carathéodory distance in  $\mathbb{N}$  and  $|\cdot|_x$  is an homogeneous norm in  $\mathbb{G}_x \mathbb{M}$ .

For us, it is convenient to regard some neighborhood of a point  $g$  as a subspace of the metric space  $(\mathbb{M}, d_c)$  and as a neighborhood of unity of the local Carnot group  $\mathcal{G}^g \mathbb{M}$  with Carnot–Carathéodory metric  $d_c^g$  (see Definition 1.2). In the sense explained below,  $\exp^{-1} : \mathcal{G}^g \mathbb{M} \rightarrow \mathbb{G}_g \mathbb{M}$  is an isometrical monomorphism of the Lie structures. Then the last definition of

$hc$ -differentiability can be reformulated as follows. Given two Carnot manifolds  $(\mathbb{M}, d_c)$  and  $(\mathbb{N}, \tilde{d}_c)$  and a set  $E \subset \mathbb{M}$ , a mapping  $f : E \rightarrow \mathbb{N}$  is called *hc-differentiable* at a point  $g \in E$  (see the paper by S. K. Vodopyanov and A. V. Greshnov [131], and also [127, 128, 129, 130]) if there exists a horizontal homomorphism  $L : (\mathcal{G}^g \mathbb{M}, d^g) \rightarrow (\mathcal{G}^{f(g)} \mathbb{N}, d_c^{f(g)})$  of the local Carnot groups such that

$$d_c^{f(g)}(f(w), L(w)) = o(d_c^g(g, w)) \quad \text{as } E \cap \mathcal{G}^g \ni w \rightarrow g. \quad (1.0.1)$$

The given definition of  $hc$ -differentiability of mappings for Carnot manifolds can be treated as a straightforward generalization of the classical definition of differentiability. Clearly, if the Carnot manifolds are Carnot groups then this definition of  $hc$ -differentiability is equivalent to the definition of  $\mathcal{P}$ -differentiability introduced by P. Pansu in [115] for an open set  $E \subset \mathbb{G}$ . For an arbitrary  $E \subset \mathbb{G}$ , the last concept was investigated by S. K. Vodopyanov [125] and by S. K. Vodopyanov and A. D. Ukhlov [136] (see also the paper by V. Magnani [93]).

In Section 3, we introduce the notion of  $hc$ -differentiability, which is adequate to the geometry of Carnot manifold, and study its properties. Moreover, in this section, we prove the  $hc$ -differentiability of a composition of  $hc$ -differentiable mappings.

In the same section we prove the  $hc$ -differentiability of rectifiable curves. In the case of curves, the definition of the  $hc$ -differentiability is interpreted as follows: a mapping  $\mathbb{R} \supset E \ni t \mapsto \gamma(t) \in \mathbb{N}$  is  $hc$ -differentiable at a point  $s \in E$  in a Carnot manifold  $\mathbb{N}$  if the relation

$$d_c^{\gamma(s)}(\gamma(s + \tau), \exp(\tau a)(\gamma(s))) = o(\tau) \quad \text{as } \tau \rightarrow 0, s + \tau \in E, \quad (1.0.2)$$

holds, where  $a \in H_{\gamma(s)} \mathbb{N}$  ((1.0.2) agrees with (1.0.1) when  $\mathbb{M} = \mathbb{R}$ , see also [99]). On Carnot groups, relation (1.0.2) is equivalent to the  $\mathcal{P}$ -differentiability of curves in the sense of Pansu [115]. Our proof of differentiability is new even for Carnot groups. We prove step by step the  $hc$ -differentiability of the absolutely differentiable curves, the Lipschitz mappings of subsets of  $\mathbb{R}$  into  $\mathbb{M}$ , and the rectifiable curves. Here we generalize a classical result and obtain the following assertion: *the continuity of horizontal derivatives of a contact mapping defined on an open set implies its pointwise hc-differentiability* (Theorem 3.3.1).

As an important corollary to these assertion, we infer that the nilpotent tangent cone is defined by the horizontal subbundle of the Carnot manifold: *tangent cones found from different collections of basis vector fields are isomorphic as local Carnot groups* (Corollary 3.3.3). Thus, the correspondence



“local basis  $\mapsto$  nilpotent tangent cone” is functorial. In the case of  $C^\infty$ -vector fields, this result was established by A. Agrachev and A. Marigo [4], and G. A. Margulis and G. D. Mostow [100] where a coordinate-free definition of the tangent cone was given.

Main results of Section 3 are formulated in short communications by S. K. Vodopyanov [127, 128] (see some details and more general results on this subject including Rademacher–Stepanov Theorem in [129, 130]).

Section 4 is dedicated to such application of results on  $hc$ -differentiability as the sub-Riemannian analog of the coarea formula. It is well known that the coarea formula

$$\int_U \mathcal{J}_k(\varphi, x) dx = \int_{\mathbb{R}^k} dz \int_{\varphi^{-1}(z)} d\mathcal{H}^{n-k}(u), \quad (1.0.3)$$

where  $\mathcal{J}_k(\varphi, x) = \sqrt{\det(D\varphi(x)D\varphi^*(x))}$ , has many applications in analysis on Euclidean spaces. Here we assume that  $\varphi \in C^1(U, \mathbb{R}^k)$ ,  $U \subset \mathbb{R}^n$ ,  $n \geq k$ . For the first time, it was established by A. S. Kronrod [88] for the case of a function  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Next, it was generalized by H. Federer first for mappings of Riemannian manifolds  $\varphi : \mathcal{M}^n \rightarrow \mathcal{N}^k$ ,  $n \geq k$ , in [40], and then, for mappings of rectifiable sets in Euclidean spaces  $\varphi : \mathcal{M}^n \rightarrow \mathcal{N}^k$ ,  $n \geq k$ , in [41]. Next, in the paper [111], M. Ohtsuka generalized the coarea formula (1.0.3) for mappings  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $n, m \geq k$ , with  $\mathcal{H}^k$ - $\sigma$ -finite image  $\varphi(\mathbb{R}^n)$ . An infinite-dimensional analog of the coarea formula was proved by H. Airault and P. Malliavin in 1988 [10] for the case of Wiener spaces. This result can be found in the monograph by P. Malliavin [97]. See other proofs and applications of the coarea formula in the monographs by L. C. Evans and R. F. Gariepy [39], M. Giaquinta, G. Modica and J. Souček [64], F. Lin and X. Yang [90].

Formula (1.0.3) can be applied in the theory of exterior forms, currents, in minimal surfaces problems (see, for example, paper by H. Federer and W. H. Fleming [42]). Also, Stokes formula can be easily obtained by using the coarea formula (see, for instance, lecture notes by S. K. Vodopyanov [126]). Because of the development of analysis on more general structures, a natural question arise to extend the coarea formula on objects of more general geometry in comparison with Euclidean spaces, especially on metric spaces and structures on sub-Riemannian geometry. In 1999, L. Ambrosio and B. Kirchheim [11] proved the analog of the coarea formula for Lipschitz mappings defined on  $\mathcal{H}^n$ -rectifiable metric space with values in  $\mathbb{R}^k$ ,  $n \geq k$ . In 2004, this formula was proved for Lipschitz mappings defined on  $\mathcal{H}^n$ -rectifiable metric space with values in  $\mathcal{H}^k$ -rectifiable metric space,  $n \geq k$ , by M. Karmanova [83, 85]. Moreover, necessary and sufficient conditions on

the image and the preimage of a Lipschitz mapping defined on  $\mathcal{H}^n$ -rectifiable metric space with values in an *arbitrary* metric space for the validity of the coarea formula were found. Independently of this result, the level sets of such mappings were investigated, and the metric analog of Implicit Function Theorem was proved by M. Karmanova [84, 85, 86].

All the above results are connected with rectifiable metric spaces. Note that, their structure is similar to the one of Riemannian manifolds. But there are also *non-rectifiable* metric spaces which geometry is not comparable with the Riemannian one. *Carnot manifolds* are of special interest. The problem of the sub-Riemannian coarea formula is one of well-known intrinsic unsolved problems.

A Heisenberg group and a Carnot group are well-known particular cases of a Carnot manifold. In 1982, P. Pansu proved the coarea formula for functions defined on a Heisenberg group [112]. Next, in [72], J. Heinonen extended this formula to smooth functions defined on a Carnot group. In [107], R. Monti and F. Serra Cassano proved the analog of the coarea formula for  $BV$ -functions defined on a two-step Carnot–Carathéodory space. One more result concerning the analogue of (1.0.3) belongs to V. Magnani. In 2000, he proved a *coarea inequality* for mappings of Carnot groups [95]. The equality was proved only for the case of a mapping defined on a Heisenberg group with values in Euclidean space  $\mathbb{R}^k$  [96]. Until now, the question about the validity of coarea formula even for a model case of a mapping of Carnot groups was open.

Main results of Section 4 are formulated in [134].

## 2 Geometry of Carnot–Carathéodory Spaces

### 2.1 Preliminary Results

Recall the definition of a Carnot manifold.

**Definition 2.1.1** (compare with [68, 110]). Fix a connected Riemannian  $C^\infty$ -manifold  $\mathbb{M}$  of a dimension  $N$ . The manifold  $\mathbb{M}$  is called a *Carnot manifold* if, in the tangent bundle  $T\mathbb{M}$ , there exists a tangent subbundle  $H\mathbb{M}$  with a finite collection of natural numbers  $\dim H_1 < \dots < \dim H_i < \dots < \dim H_M = N$ ,  $1 < i < M$ , and each point  $p \in \mathbb{M}$  possesses a neighborhood  $U \subset \mathbb{M}$  with a collection of  $C^1$ -smooth vector fields  $X_1, \dots, X_N$  on  $U$  enjoying the following properties.

For each  $v \in U$  we have

- (1)  $X_1(v), \dots, X_N(v)$  constitutes a basis of  $T_v\mathbb{M}$ ;

(2)  $H_i(v) = \text{span}\{X_1(v), \dots, X_{\dim H_i}(v)\}$  is a subspace of  $T_v\mathbb{M}$  of a dimension  $\dim H_i$ ,  $i = 1, \dots, M$ , where  $H_1(v) = H_v\mathbb{M}$ ;

(3)

$$[X_i, X_j](v) = \sum_{\deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v) \quad (2.1.1)$$

where the *degree*  $\deg X_k$  equals  $\min\{m \mid X_k \in H_m\}$ ;

(4) a quotient mapping  $[\cdot, \cdot]_0 : H_1 \times H_j/H_{j-1} \mapsto H_{j+1}/H_j$  induced by Lie brackets, is an epimorphism for all  $1 \leq j < M$ .

**Remark 2.1.2.** Note [110] that Condition 4 is necessary only for obtaining results of Subsections 2.8 and 3.3 and its corollaries, and Section 4. The point is that in statements of these subsections, we use the fact that any two points of a local Carnot group (see the definition below) can be joined by a horizontal (with respect to the local Carnot group) curve that consists of at most  $L$  segments of integral lines of horizontal (with respect to the local Carnot group) vector fields. The latter is impossible without Condition 4.

**Remark 2.1.3.** Consider a  $C^2$ -smooth local diffeomorphism  $\eta : U \rightarrow \mathbb{R}^N$ ,  $U \subset \mathbb{M}$ . Then  $\eta_* X_i = D\eta \langle X_i \rangle$  are also  $C^1$ -vector fields,  $i = 1, \dots, N$ . We have the following relations instead of (2.1.1):

$$\eta_*[X_i, X_j](w) = [\eta_* X_i, \eta_* X_j](w) = \sum_{\deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(\eta^{-1}(w)) \eta_* X_k(w).$$

Denote by  $X(w)$  the matrix, the  $i$ th column of which consists of the coordinates of  $\eta_* X_i(w)$  in the standard basis  $\{\partial_j\}_{j=1}^N$ . Then the entries of  $X(w)$  are  $C^1$ -functions. Note that

$$\eta_*[X_i, X_j](w) = X(w)(c_{ij1}(\eta^{-1}(w)), \dots, c_{ijN}(\eta^{-1}(w)))^T.$$

Consequently,

$$(c_{ij1}(\eta^{-1}(w)), \dots, c_{ijN}(\eta^{-1}(w)))^T = (X(w))^{-1} \cdot \eta_*[X_i, X_j](w).$$

From here it follows that  $c_{ijk} \circ \eta^{-1}$  are continuous,  $k = 1, \dots, N$ . Since  $\eta$  is continuous, then we have that each  $c_{ijk} = (c_{ijk} \circ \eta^{-1}) \circ \eta$  is also continuous,  $k = 1, \dots, N$ .

**Example** (A Carnot Manifold with  $C^1$ -Smooth Vector Fields). Consider the  $C^M$  vector fields  $X_1, \dots, X_n \in H$ . Choose a basis in  $H_2 = \text{span}\{H, [H, H]\}$  by the following way:

$$X_k(v) = \sum_{i,j} a_{ij}^k(v) [X_i, X_j](v) + \sum_l b_l^k(v) X_l,$$

where  $a_{ij}^k(v), b_l^k(v) \in C^1$ ,  $i, j, l = 1, \dots, N$ ,  $k = n + 1, \dots, \dim H_2$ . Similarly, we choose the following basis in  $H_{m+1} = \text{span}\{H_m, [H, H_m]\}$ ,  $m = 2, \dots, M - 1$ :

$$X_k(v) = \sum_{i,j} a_{ij}^k(v)[X_i, X_j](v) + \sum_l b_l^k(v)X_l,$$

where  $a_{ij}^k(v), b_l^k(v) \in C^1$ ,  $i = 1, \dots, N$ ,  $j, l = \dim H_{m-1} + 1, \dots, \dim H_m$ ,  $k = \dim H_m + 1, \dots, \dim H_{m+1}$ .

**Assumption 2.1.4.** Throughout the paper, we assume that all the basis vector fields  $X_1, \dots, X_N$  are  $C^{1,\alpha}$ -smooth, and, consequently, their commutators are  $H^\alpha$ -continuous,  $\alpha \in [0, 1]$ .

In some parts of this paper, we consider cases when the derivatives of the basis vector fields are  $H^\alpha$ -continuous with respect to some nonnegative symmetric function  $\mathfrak{d} : U \times U \rightarrow \mathbb{R}$ ,  $U \subseteq \mathbb{M}$ , such that  $\mathfrak{d} \geq C\rho$ ,  $0 < C < \infty$ , where  $C$  depends only on  $U$ , and  $\rho$  is Riemannian distance. Some additional properties of  $\mathfrak{d}$  are described below when it is necessary.

**Notation 2.1.5.** In the paper:

1. The symbol  $X \in C^{1,0}$  means that  $X \in C^1$ , and the symbol  $X \in C^0$  means  $X \in C$ .
2. 0-Hölder continuity means the ordinary continuity. We denote a modulus of continuity of a mapping  $f$  by  $\omega_f(\delta)$ .
3. The Riemannian distance is denoted by the symbol  $\rho$ .

**Theorem 2.1.6.** *The coefficients  $\bar{c}_{ijk} = c_{ijk}(u)$  of (2.1.1) with  $\deg X_i + \deg X_j = \deg X_k$  define a graded nilpotent Lie algebra.*

*Proof.* Fix an arbitrary point  $u \in \mathbb{M}$  and show that the collection  $\{c_{ijk}(u)\}$  with  $\deg X_k = \deg X_i + \deg X_j$ , enjoy the Jacobi identity and, thus, define the structure of a Lie algebra. The property  $\bar{c}_{ijk} = -\bar{c}_{jik}$  is evident. Prove that the collection  $\{\bar{c}_{ijk}\}$  under consideration enjoys Jacobi identity.

**1<sup>ST</sup> STEP.** We may assume without loss of generality that  $X_1, \dots, X_N$  are the vector fields on an open set of  $\mathbb{R}^N$  (otherwise, consider the local  $C^2$ -diffeomorphism  $\eta$  similarly to Remark 2.1.3).

For a vector field  $X_i(x) = \sum_{j=1}^N \eta_{ij}(x)\partial_j$ , consider the mollification  $(X_i)_h(x) = \sum_{j=1}^N (\eta_{ij} * \omega_h)(x)\partial_j$ ,  $i = 1, \dots, N$ , where the function  $\omega \in C_0^\infty(B(0, 1))$  is such

that  $\int_{B(0,1)} \omega(x) dx = 1$ , and  $\omega_h(x) = \frac{1}{h^N} \omega\left(\frac{x}{h}\right)$ . By the properties of mol-

lification  $\eta_{ij} * \omega_h$ ,  $i, j = 1, \dots, N$ , we have  $(X_i)_h \xrightarrow[h \rightarrow 0]{C^1} X_i$  locally in some neighborhood of  $u$ . Note that the vector fields  $(X_i)_h(v)$ ,  $i = 1, \dots, N$ , meet the Jacobi identity, and are a basis of  $T_v \mathbb{M}$  for  $v$  belonging to some neighborhood of  $u$ , if the parameter  $h$  is small enough. Consequently, setting  $[(X_i)_h, (X_j)_h] = \sum_{k=1}^N (c_{ijk})_h (X_k)_h$ , we have

$$\begin{aligned} & \sum_k \sum_l (c_{ijk})_h (c_{kml})_h (X_l)_h + \sum_k \sum_l (c_{mik})_h (c_{kjl})_h (X_l)_h \\ & + \sum_k \sum_l (c_{jmk})_h (c_{kil})_h (X_l)_h - \sum_l [(X_m)_h (c_{ijl})_h] (X_l)_h \\ & - \sum_l [(X_j)_h (c_{mil})_h] (X_l)_h - \sum_l [(X_i)_h (c_{jml})_h] (X_l)_h = 0. \end{aligned}$$

Note that, since  $(X_i)_h \xrightarrow[h \rightarrow 0]{C^1} X_i$  locally, and the vector fields  $\{(X_i)_h\}_{i=1}^N$  are linearly independent for all  $h \geq 0$  small enough, we have  $(c_{ijk})_h \rightarrow c_{ijk}$  as  $h \rightarrow 0$ .

Now, fix  $1 \leq l \leq N$ . Since the vector fields  $\{(X_i)_h\}_{i=1}^N$  are linearly independent for  $h > 0$  small enough, we have

$$\begin{aligned} & \sum_k (c_{ijk})_h (c_{kml})_h + \sum_k (c_{mik})_h (c_{kjl})_h + \sum_k (c_{jmk})_h (c_{kil})_h \\ & - [(X_m)_h (c_{ijl})_h] - [(X_j)_h (c_{mil})_h] - [(X_i)_h (c_{jml})_h] = 0 \quad (2.1.2) \end{aligned}$$

for each fixed  $l$  in some neighbourhood of  $u$ . Fix  $i, j, m$  and  $l$  such that  $\deg X_l = \deg X_i + \deg X_j + \deg X_m$ , and consider a test function  $\varphi \in C_0^\infty(U)$  on some small compact neighborhood  $U \ni u$ ,  $U \Subset \mathbb{M}$ . We multiply both sides of (2.1.2) on  $\varphi$  and integrate the result over  $U$ . For  $h > 0$  small enough, we have

$$\begin{aligned} 0 &= \int_U \left[ \sum_k (c_{ijk})_h(v) (c_{kml})_h(v) + \sum_k (c_{mik})_h(v) (c_{kjl})_h(v) \right. \\ & \quad \left. + \sum_k (c_{jmk})_h(v) (c_{kil})_h(v) \right] \cdot \varphi(v) dv - \int_U [(X_m)_h (c_{ijl})_h](v) \cdot \varphi(v) dv \\ & \quad - \int_U [(X_j)_h (c_{mil})_h](v) \cdot \varphi(v) dv - \int_U [(X_i)_h (c_{jml})_h](v) \cdot \varphi(v) dv. \end{aligned}$$

Show that, among the last three integrals, the first one tends to zero as  $h \rightarrow 0$ . Indeed,

$$\int_U [(X_m)_h(c_{ijl})_h](v) \cdot \varphi(v) dv = - \int_U [(X_m)_h^* \varphi](v) \cdot (c_{ijl})_h(v) dv,$$

where  $(X_i)_h^*$  is an adjoint operator to  $(X_i)_h$ . The right-hand part integral tends to zero as  $h \rightarrow 0$  since the value  $[(X_m)_h^* \varphi](v)$  is uniformly bounded in  $U$  as  $h \rightarrow 0$ , and  $(c_{ijl})_h(v) \rightarrow 0$  as  $h \rightarrow 0$  in view of the choice of  $l$ . The similar conclusion is true regarding the last two integrals.

Consequently, taking into account the facts that  $(c_{ijk})_h \rightarrow c_{ijk}$  locally, and  $c_{ijk} = 0$  for  $\deg X_k > \deg X_i + \deg X_j$ , and using du Bois–Reymond Lemma for  $h \rightarrow 0$  we infer

$$\begin{aligned} \sum_{k: \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) c_{kml}(v) + \sum_{k: \deg X_k \leq \deg X_m + \deg X_i} c_{mik}(v) c_{kjl}(v) \\ + \sum_{k: \deg X_k \leq \deg X_j + \deg X_m} c_{jmk}(v) c_{kil}(v) = 0 \quad (2.1.3) \end{aligned}$$

for all  $v \in \mathbb{M}$  close enough to  $u$ .

**2<sup>ND</sup> STEP.** For fixed  $l$ , such that  $\deg X_l = \deg X_i + \deg X_j + \deg X_m$ , investigate the properties of the index  $k$ . Consider the first sum. Since  $\deg X_l \leq \deg X_k + \deg X_m$ , we have  $\deg X_k \geq \deg X_l - \deg X_m = \deg X_i + \deg X_j$ . By (2.1.1),  $\deg X_k \leq \deg X_i + \deg X_j$ , and, consequently,  $\deg X_k = \deg X_i + \deg X_j$ . The other two cases are considered similarly. Thus, the sum (2.1.3) with  $\deg X_l = \deg X_i + \deg X_j + \deg X_m$  and  $v = u$  is

$$\begin{aligned} \sum_{\deg X_k = \deg X_i + \deg X_j} c_{ijk}(u) c_{kml}(u) + \sum_{\deg X_k = \deg X_m + \deg X_i} c_{mik}(u) c_{kjl}(u) \\ + \sum_{\deg X_k = \deg X_j + \deg X_m} c_{jmk}(u) c_{kil}(u) = 0. \end{aligned}$$

The coefficients  $\{\bar{c}_{ijk} = c_{ijk}(u)\}_{\deg X_k = \deg X_i + \deg X_j}$  enjoy the Jacobi identity, and, thus, they define the Lie algebra. The theorem follows.  $\square$

We construct the Lie algebra  $\mathfrak{g}^u$  from Theorem 2.1.6 as a graded nilpotent Lie algebra of vector fields  $\{(\widehat{X}_i^u)'\}_{i=1}^N$  on  $\mathbb{R}^N$  [117]. Thus the relation

$$[(\widehat{X}_i^u)', (\widehat{X}_j^u)'] = \sum_{\deg X_k = \deg X_i + \deg X_j} c_{ijk}(u) (\widehat{X}_k^u)'$$

holds for the vector fields  $\{(\widehat{X}_i^u)'\}_{i=1}^N$  everywhere on  $\mathbb{R}^N$ .

**Notation 2.1.7.** We use the following standard notations: for each  $N$ -dimensional multi-index  $\mu = (\mu_1, \dots, \mu_N)$ , its *homogeneous norm* equals  $|\mu|_h = \sum_{i=1}^N \mu_i \deg X_i$ .

**Definition 2.1.8.** The Carnot group  $\mathbb{G}_u \mathbb{M}$  corresponding to the Lie algebra  $\mathfrak{g}^u$ , is called the *nilpotent tangent cone* of  $\mathbb{M}$  at  $u \in \mathbb{M}$ . We construct  $\mathbb{G}_u \mathbb{M}$  in  $\mathbb{R}^N$  as a group algebra [117]. By Campbell–Hausdorff formula, the group operation is defined for the basis vector fields  $(\widehat{X}_i^u)'$  on  $\mathbb{R}^N$ ,  $i = 1, \dots, N$ , to be left-invariant [117]: if

$$x = \exp\left(\sum_{i=1}^N x_i (\widehat{X}_i^u)'\right), \quad y = \exp\left(\sum_{i=1}^N y_i (\widehat{X}_i^u)'\right) \text{ then } x \cdot y = z = \exp\left(\sum_{i=1}^N z_i (\widehat{X}_i^u)'\right),$$

where

$$\begin{aligned} z_i &= x_i + y_i, \quad \deg X_i = 1, \\ z_i &= x_i + y_i + \sum_{\substack{|e_l + e_j|_h = 2, \\ l < j}} F_{e_l, e_j}^i(u) (x_l y_j - y_l x_j), \quad \deg X_i = 2, \\ z_i &= x_i + y_i + \sum_{\substack{|\mu + \beta|_h = k, \\ \mu > 0, \beta > 0}} F_{\mu, \beta}^i(u) x^\mu \cdot y^\beta \\ &= x_i + y_i + \sum_{\substack{|\mu + e_l + \beta + e_j|_h = k, \\ l < j}} G_{\mu, \beta, l, j}^i(u) x^\mu y^\beta (x_l y_j - y_l x_j), \quad \deg X_i = k. \end{aligned} \tag{2.1.4}$$

Theorem 2.1.6 implies

**Theorem 2.1.9** ([46]). *If  $\{\frac{\partial}{\partial x_l}\}_{l=1}^N$  is the standard basis in  $\mathbb{R}^N$  then the  $j$ -th coordinate of a vector field  $(\widehat{X}_i^u)'(x) = \sum_{j=1}^N z_i^j(u, x) \frac{\partial}{\partial x_j}$  can be written as*

$$z_i^j(u, x) = \begin{cases} \delta_{ij} & \text{if } j \leq \dim H_{\deg X_i}, \\ \sum_{\substack{|\mu + e_i|_h = \deg X_j, \\ \mu > 0}} F_{\mu, e_i}^j(u) x^\mu & \text{if } j > \dim H_{\deg X_i}. \end{cases} \tag{2.1.5}$$

**Definition 2.1.10.** Suppose that  $u \in \mathbb{M}$  and  $(v_1, \dots, v_N) \in B_E(0, r)$  where  $B_E(0, r)$  is an Euclidean ball in  $\mathbb{R}^N$ . Define a mapping  $\theta_u(v_1, \dots, v_N) : B_E(0, r) \rightarrow \mathbb{M}$  as follows:

$$\theta_u(v_1, \dots, v_N) = \exp\left(\sum_{i=1}^N v_i X_i\right)(u).$$

It is known, that  $\theta_u$  is a  $C^1$ -diffeomorphism if  $0 < r \leq r_u$  for some  $r_u > 0$ . The collection  $\{v_i\}_{i=1}^N$  is called *the normal coordinates* or *the coordinates of the 1<sup>st</sup> kind* (with respect to  $u \in \mathbb{M}$ ) of the point  $v = \theta_u(v_1, \dots, v_N)$ .

**Assumption 2.1.11.** The compactly embedded neighborhood  $U \subset \mathbb{M}$  under consideration is such that  $\theta_u(B_E(0, r_u)) \supset U$  for all  $u \in U$ .

By means of the exponential map we can push-forward the vector fields  $(\hat{X}_i^u)'$  onto  $U$  for obtaining the vector fields  $\hat{X}_i^u = (\theta_u)_*(\hat{X}_i^u)'$  where

$$(\theta_u)_*\langle Y \rangle(\theta_u(x)) = D\theta_u(x)\langle Y \rangle,$$

$Y \in T_x \mathbb{R}^N$ . Note that  $\hat{X}_i^u(u) = X_i(u)$ . Indeed, on the one hand, by the definition, we have  $(\theta_u)_*^{-1}X_i(0) = e_i$ . On the other hand, Theorem 2.1.9 implies  $(\hat{X}_i^u)'(0) = e_i$ . Thus  $\hat{X}_i^u(u) = X_i(u)$ .

**Theorem 2.1.12.** *The vector fields  $\hat{X}_i^u$ ,  $i = 1, \dots, N$ , are locally  $H^\alpha$ -continuous on  $u$ .*

The proofs of this theorem and of many other assertions concerning smoothness use often the following lemma (see its proof in Section 5).

**Theorem 2.1.13.** *Consider the ODE*

$$\begin{cases} \frac{dy}{dt} = f(y, v, u), \\ y(0) = 0 \end{cases} \quad (2.1.6)$$

where  $t \in [0, 1]$ ,  $y, v, u \in W \subset \mathbb{R}^N$  and  $\text{Lip}_y(f) = L < 1$ .

1. *If the mapping  $f(y, v, u) = f(y, u) \in C^1(y) \cap C^\alpha(u)$  then the solution  $y(t, u) \in C^\alpha(u)$  locally.*
2. *If  $f(y, v, u) \in C^{1,\alpha}(y, u) \cap C^1(v)$  and  $\frac{\partial f}{\partial v} \in C^{1,\alpha}(y, u)$  then  $\frac{dy(t, v, u)}{dv} \in C^\alpha(u)$  locally.*

**Remark 2.1.14.** The following statements are proved similarly to Theorem 2.1.13.

1. *If the mapping  $f(y, v, u)$  from (2.1.6) does not depend on  $v$ , and it is  $C^1$ -smooth in  $y$  and it is locally  $\alpha$ -Hölder with respect to nonnegative symmetric function  $\mathfrak{d}$  defined on  $U \times U$ ,  $U \Subset \mathbb{M}$ , such that  $\mathfrak{d} \geq C\rho$ ,  $0 < C < \infty$ , where  $C$  depends only on  $U$ , then the solution  $y(t, u)$  is also locally  $\alpha$ -Hölder with respect to  $\mathfrak{d}$ .*



2. If  $f(y, v, u) \in C^1(y, u) \cap C^1(v)$ , its derivatives in  $y$  and in  $u$  are locally  $\alpha$ -Hölder with respect to  $\mathfrak{d}$ ,  $\frac{\partial f}{\partial v} \in C^1(y, u)$ , and the derivatives of  $\frac{\partial f}{\partial v}$  in  $y$  and in  $u$  are locally  $\alpha$ -Hölder with respect to  $\mathfrak{d}$  then  $\frac{dy(t, v, u)}{dv}$  is locally  $\alpha$ -Hölder with respect to  $\mathfrak{d}$ .

**Remark 2.1.15.** One of particular cases of  $\mathfrak{d}$  is  $d_\infty$ .

**Notation 2.1.16.** Hereinafter, we denote a nonnegative symmetric function defined on  $U \times U$ ,  $U \subseteq \mathbb{M}$ , possessing properties from item 1 of Remark 2.1.14, by  $\mathfrak{d}$ .

*Proof of Proposition 2.1.12.* 1<sup>ST</sup> STEP. Taking into account Assumption 2.1.4, we have the table

$$[(\hat{X}_i^u)', (\hat{X}_j^u)'](v) = \sum_{\deg X_k = \deg X_i + \deg X_j} c_{ijk}(u) (\hat{X}_k^u)'(v).$$

By means of Assumption 2.1.4 and Definition 2.1.1, the functions  $c_{ijk}(u)$  from (2.1.1) are  $H^\alpha$ -continuous.

If  $X = \sum_{i=1}^N x_i (\hat{X}_i^u)'$  and  $Y = \sum_{i=1}^N y_i (\hat{X}_i^u)'$  then by Campbell–Hausdorff formula we have  $\exp tY \circ \exp tX(g) = \exp Z(t)(g)$  where  $Z(t) = tZ_1 + t^2Z_2 + \dots + t^M Z_M$  and  $Z_1, Z_2, \dots$  are some vector fields independent of  $t$ . Dynkin formula (see, for instance, [117]) for calculating  $Z_l(t)$ ,  $1 \leq l \leq M$ , gives

$$\begin{aligned} Z_l &= \frac{1}{n} \sum_{k=1}^l \frac{(-1)^{k-1}}{k} \sum_{(p)(q)} \frac{(\text{ad } Y)^{q_k} (\text{ad } X)^{p_k} \dots (\text{ad } Y)^{q_1} (\text{ad } X)^{p_1-1} X}{p_1! q_1! \dots p_k! q_k!} \\ &= \sum_{(p)(q)} C_{(p)(q)} (\text{ad } Y)^{q_k} (\text{ad } X)^{p_k} \dots (\text{ad } Y)^{q_1} (\text{ad } X)^{p_1-1} X, \end{aligned}$$

where  $C_{(p)(q)} = \text{const}$ ,  $(p) = (p_1, \dots, p_k)$ ,  $(q) = (q_1, \dots, q_k)$ . We sum over all natural  $p_1, q_1, \dots, p_k, q_k$ , such that  $p_i + q_i > 0$ ,  $p_1 + q_1 + \dots + p_k + q_k = l$ , and  $(\text{ad } A)B = [A, B]$ ,  $(\text{ad } A)^0 B = B$ . The each summand can be represented as a sum

$$Z_l(v) = \sum_{i=1}^N d_{j,l}(u, x, y) (\hat{X}_j^u)'(v),$$

where  $d_{j,l}(u, x, y)$  are polynomial functions of  $x = (x_1, \dots, x_N)$ ,  $y = (y_1, \dots, y_N)$  coefficients of which are polynomial functions of  $\{c_{lmk}(u)\}$  and, consequently, are Hölder in  $u$ . More exactly,

$$\sum_{l=2}^M Z_l = \sum_{l=2}^M \sum_{j=1}^N d_{j,l}(u, x, y) (\hat{X}_j^u)' = \sum_{j=1}^N \left[ \sum_{l=2}^M \sum_{\substack{|\mu+\beta|_h=l, \\ \mu>0, \beta>0}} F_{\mu,\beta}^j(u) x^\mu \cdot y^\beta \right] (\hat{X}_j^u)'.$$

Consequently,

$$d_{j,l}(u, x, y) = \sum_{l=2}^M \sum_{\substack{|\mu+\beta|_h=l, \\ \mu>0, \beta>0}} F_{\mu,\beta}^j(u) x^\mu \cdot y^\beta.$$

Hence,  $F_{\mu,\beta}^j(u)$  are  $H^\alpha$ -continuous in  $u$ , and  $(\widehat{X}_i^u)'$  are also  $H^\alpha$ -continuous on  $u$  (see (2.1.5)).

2<sup>ND</sup> STEP. Consider the following Cauchy problem:

$$\begin{cases} \frac{d\Phi(t, u, \xi)}{dt} = \sum_{i=1}^N \xi_i X_i(\Phi), \\ \Phi(0, u, \xi) = u, \end{cases} \quad (2.1.7)$$

where  $\xi = (\xi_1, \dots, \xi_N)$ . Note that  $\Phi(t, u, \xi) = \exp\left(\sum_{i=1}^N t \xi_i X_i\right)(u)$ . We can assume without loss of generality, that  $\mathbb{M} = \mathbb{R}^N$ . If Assumption 2.1.4 holds then the mapping  $f(\xi, \Phi) = \sum_{i=1}^N \xi_i X_i(\Phi)$  is  $C^{1,\alpha}$ -smooth in  $\xi$  and  $\Phi$ . From the definition, it follows, that  $\theta_u(\xi) = \Phi(1, u, \xi)$ .

By theorem 2.1.13 on smooth dependence of ODE solution on parameters (see Section 5 for details), it is easy to see, that the differential  $D\theta_u(y)$  is  $H^\alpha$ -continuous in  $u$ .

Since  $\widehat{X}_i^u(x) = D\theta_u(y)(\widehat{X}_i^u)'(y)$ ,  $x = \theta_u(y)$ , the proposition follows from results of the 1<sup>ST</sup> and 2<sup>ND</sup> steps.  $\square$

**Remark 2.1.17.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , then  $\widehat{X}_i^u$ ,  $i = 1, \dots, N$ , are locally Hölder on  $u$  with respect to  $\mathfrak{d}$ .

**Definition 2.1.18.** The local Lie group corresponding to the Lie algebra  $\{\widehat{X}_i^u\}_{i=1}^N$ , is called the *local Carnot group*  $\mathcal{G}^u \mathbb{M}$  at  $u \in \mathbb{M}$ . Define it in such a way that the mapping  $\theta_u$  is a *group isomorphism* of some neighborhood of the unity of the group  $\mathbb{G}_u \mathbb{M}$  and  $\mathcal{G}^u \mathbb{M}$ . The canonical Riemannian structure is defined by scalar product at the unit of  $\mathcal{G}^u \mathbb{M}$  coinciding with those in  $T_u \mathbb{M}$ .

**Remark 2.1.19.** Recall that the vector fields  $\widehat{X}_i^u$ ,  $i = 1, \dots, N$ , are locally  $H^\alpha$ -continuous on  $\mathbb{M}$ ,  $\alpha \in [0, 1]$ . The exponential mapping  $\exp\left(\sum_{i=1}^N a_i \widehat{X}_i^u\right)(g)$  is not defined correctly for such fields. Therefore, in view of smoothness of  $(\theta_u^{-1})_*(\widehat{X}_i^u)$ ,  $i = 1, \dots, N$ , we define the point

$$a = \exp\left(\sum_{i=1}^N a_i \widehat{X}_i^u\right)(g)$$

according to Definition 2.1.18: first, we obtain a point

$$a_u = \exp\left(\sum_{i=1}^N a_i \cdot (\theta_u^{-1})_*(\widehat{X}_i^u)\right)(\theta_u^{-1}(g)),$$

and then we define  $a = \theta_u(a_u)$ . Moreover, we similarly define the whole curve corresponding to this exponential mapping. Suppose that

$$\begin{cases} \dot{\gamma}_u(t) = \sum_{i=1}^N a_i \cdot (\theta_u^{-1})_*(\widehat{X}_i^u)(\gamma_u(t)) \\ \gamma_u(0) = \theta_u^{-1}(g). \end{cases}$$

Then, for the curve  $\gamma(t) = \theta_u(\gamma_u(t))$ , we have

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^N a_i \widehat{X}_i^u(\gamma(t)) \\ \gamma(0) = g. \end{cases}$$

In particular, we have:

1. The exponential mapping  $\widehat{\theta}_u(v_1, \dots, v_n) = \exp\left(\sum_{i=1}^N v_i \widehat{X}_i^u\right)(u)$  is defined as

$$\theta_u\left[\exp\left(\sum_{i=1}^N v_i (\widehat{X}_i^u)'\right)(0)\right];$$

and the mapping  $\widehat{\theta}_u^w(v_1, \dots, v_n) = \exp\left(\sum_{i=1}^N v_i \widehat{X}_i^u\right)(w)$  is defined as

$$\theta_u\left[\exp\left(\sum_{i=1}^N v_i (\widehat{X}_i^u)'\right)(\theta_u^{-1}(w))\right].$$

2. The inverse mapping  $\exp^{-1}$  is also defined by the unique way for vector fields  $\{\widehat{X}_i^u\}_{i=1}^N$  since it is defined by the unique way for  $\{(\widehat{X}_i^u)'\}_{i=1}^N$ .
3. The group operation is defined by the following way: if  $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)$ ,  $y = \exp\left(\sum_{i=1}^N y_i \widehat{X}_i^u\right)$  then  $x \cdot y = \exp\left(\sum_{i=1}^N y_i \widehat{X}_i^u\right) \circ \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right) = \exp\left(\sum_{i=1}^N z_i \widehat{X}_i^u\right)$  where  $z_i$  are taken from Definition 2.1.8.

4. Using the normal coordinates  $\widehat{\theta}_u^{-1}$ , define the action of the *dilation group*  $\delta_\varepsilon^u$  in the local Carnot group  $\mathcal{G}^u\mathbb{M}$ : to an element  $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(u)$ , assign  $\delta_\varepsilon^u x = \exp\left(\sum_{i=1}^N x_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(u)$  in the cases where the right-hand side makes sense.

**Property 2.1.20.** For each vector field  $\widehat{X}_i^u, i = 1, \dots, N$ , we have  $(\delta_\varepsilon^u)_* \widehat{X}_i^u(g) = \varepsilon^{\deg X_i} \widehat{X}_i^u(\delta_\varepsilon^u g)$ .

This property comes from those on the “canonical” Carnot group  $T_u\mathbb{M}$  [46].

**Lemma 2.1.21** ([127]). *Suppose that  $u \in U$ . The equality*

$$\sum_{i=1}^j \sum_{\substack{|\mu+e_i|_h = \deg X_j, \\ |\mu+e_i|=l, \mu>0}} x_i F_{\mu,e_i}^j(u) x^\mu = 0, \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N,$$

*holds for all  $\deg X_j \geq 2, l = 2, \dots, \deg X_j$ .*

*Proof.* Consider a vector field  $X = \sum_{i=1}^N x_i (\widehat{X}_i^u)'$ . It is known that  $\exp rsX \circ \exp rtX(g) = \exp r(s+t)X(g)$ . Therefore, by (2.1.4), we have

$$\sum_{\substack{|\mu+\beta|_h = \deg X_j, \\ \mu>0, \beta>0}} r^{|\mu+\beta|} F_{\mu,\beta}^j(g) s^{|\mu|} x^\mu \cdot t^{|\beta|} x^\beta = 0$$

for all fixed  $s$  and  $t$ ,  $\deg X_j \geq 2$ . It follows that the coefficients at all powers of  $r$  vanish. In particular, if  $|\mu + \beta| = l \geq 2$  then

$$\sum_{\substack{|\mu+\beta|_h = \deg X_j, \\ \mu>0, \beta>0, |\mu+\beta|=l}} F_{\mu,\beta}^j(g) s^{|\mu|} x^\mu \cdot t^{|\beta|} x^\beta = 0.$$

Consequently, if  $|\beta| = 1$  then we infer

$$P(s) = \sum_{l=2}^{\deg X_j} s^{l-1} \sum_{i=1}^j \sum_{\substack{|\mu+e_i|_h = \deg X_j, \\ |\mu+e_i|=l, \mu>0}} x_i F_{\mu,e_i}^j(g) x^\mu \equiv 0,$$

where  $s$  is an arbitrarily small parameter. Therefore, all coefficients of the polynomial  $P(s)$  at the powers of  $s$  vanish. The lemma follows.  $\square$

**Lemma 2.1.22** ([127]). *Let  $u \in U$  be an arbitrary point. Then*

$$a = \exp\left(\sum_{i=1}^N a_i X_i\right)(u) = \exp\left(\sum_{i=1}^N a_i \widehat{X}_i^u\right)(u)$$

for all  $|a_i| < r_u$ ,  $i = 1, \dots, N$ .

*Proof.* Lemma 2.1.21 implies that the line  $\mathbb{R} \ni t \mapsto t(a_1, \dots, a_N)$  is the integral line of the vector field  $\sum_{i=1}^N a_i (\widehat{X}_i^u)'$  starting at 0 as  $t = 0$ . By the definition of the exponential map, we infer  $\mathbb{R}^N \ni (a_1, \dots, a_N) = \sum_{i=1}^N a_i (\widehat{X}_i^u)'(a_1, \dots, a_N) = \exp\left(\sum_{i=1}^N a_i (\widehat{X}_i^u)'\right)$ , i. e. the exponential map equals the identity. From this, it follows immediately that

$$\begin{aligned} a = \theta_u(a_1, \dots, a_N) &= \theta_u\left(\sum_{i=1}^N a_i (\widehat{X}_i^u)'\right) \\ &= \theta_u\left(\exp\left(\sum_{i=1}^N a_i (\widehat{X}_i^u)'\right)\right) = \exp\left(\sum_{i=1}^N a_i \widehat{X}_i^u\right) \end{aligned}$$

according to Remark 2.1.19.  $\square$

**Definition 2.1.23.** Suppose that  $\mathbb{M}$  is a Carnot manifold, and  $u \in \mathbb{M}$ . For  $a, p \in \mathcal{G}^u \mathbb{M}$ , where

$$a = \exp\left(\sum_{i=1}^N a_i \widehat{X}_i^u\right)(p),$$

we define the quasimetric  $d_\infty^u(a, p) = \max_i \{|a_i|^{\frac{1}{\deg X_i}}\}$  on  $\mathcal{G}^u \mathbb{M}$ .

The following properties comes from those on the “canonical” Carnot group  $T_u \mathbb{M}$  [46].

**Property 2.1.24.** It is easy to see that  $d_\infty^u(x, y)$  is a quasimetric on  $\mathcal{G}^u \mathbb{M}$  meeting the following properties:

1.  $d_\infty^u(x, y) \geq 0$ ,  $d_\infty^u(x, y) = 0$  if and only if  $x = y$ ;
2.  $d_\infty^u(u, v) = d_\infty^u(v, u)$ ;
3. the quasimetric  $d_\infty^u(x, y)$  is continuous with respect to each of its variables;

4. there exists a constant  $Q_\Delta = C_\Delta(U)$  such that the inequality

$$d_\infty^u(x, y) \leq Q_\Delta(d_\infty^u(x, z) + d_\infty^u(z, y))$$

holds for every triple of points  $x, y, z \in U$ .

**Property 2.1.25.** Let

$$w_\varepsilon = \exp\left(\sum_{i=1}^N \varepsilon^{\deg X_i} w_i \widehat{X}_i^u\right)(v) \text{ and } g_\varepsilon = \exp\left(\sum_{i=1}^N \varepsilon^{\deg X_i} g_i \widehat{X}_i^u\right)(v).$$

Then  $d_\infty^u(w_\varepsilon, g_\varepsilon) = \varepsilon d_\infty^u(w_1, g_1)$ .

By  $\text{Box}^u(x, r)$  we denote a set  $\{y \in \mathbb{M} : d_\infty^u(x, y) < r\}$ .

**Property 2.1.26.** We have  $\delta_\varepsilon^u(\text{Box}^u(u, r)) = \text{Box}^u(u, \varepsilon r)$ .

## 2.2 Gromov's Theorem on the Nilpotentization of Vector Fields and Estimate of the Diameter of a Box

**Definition 2.2.1.** Suppose that  $\mathbb{M}$  is a Carnot manifold, and let  $U \subset \mathbb{M}$  be as in Assumption 2.1.11. Given

$$v = \exp\left(\sum_{i=1}^N v_i X_i\right)(u)$$

$u, v \in U$ , define the quasimetric  $d_\infty(u, v) = \max_i \{|v_i|^{\frac{1}{\deg X_i}}\}$ . By  $\text{Box}(x, r)$  we denote a set  $\{y \in \mathbb{M} : d_\infty(x, y) < r\}$ ,  $r \leq r_x$ .

**Definition 2.2.2.** Using the normal coordinates  $\theta_u^{-1}$ , define the action of the *dilation group*  $\Delta_\varepsilon^u$  in a neighborhood of a point  $u \in \mathbb{M}$ : to an element  $x = \exp\left(\sum_{i=1}^N x_i X_i\right)(u)$ , assign  $\Delta_\varepsilon^u x = \exp\left(\sum_{i=1}^N x_i \varepsilon^{\deg X_i} X_i\right)(u)$  in the cases where the right-hand side makes sense.

**Property 2.2.3.** By Lemma 2.1.22 we have  $\Delta_\varepsilon^u x = \delta_\varepsilon^u x$ .

**Property 2.2.4.** By Lemma 2.1.22 we have  $\text{Box}^u(u, r) = \text{Box}(u, r)$ .

**Property 2.2.5.** We have  $\Delta_\varepsilon^u(\text{Box}(u, r)) = \text{Box}(u, \varepsilon r)$ ,  $r \in (0, r_u]$ .

**Property 2.2.6.** The quasimetric  $d_\infty$  has the following properties:

1.  $d_\infty(u, v) \geq 0$ ,  $d_\infty(u, v) = 0$  if and only if  $u = v$ ;
2.  $d_\infty(u, v) = d_\infty(v, u)$ ;

3. the quasimetric  $d_\infty(u, v)$  is continuous with respect to each of its variables;
4. there exists a constant  $Q = Q(U)$  such that the inequality

$$d_\infty(u, v) \leq Q(d_\infty(u, w) + d_\infty(w, v))$$

holds for every triple of points  $u, w, v \in U$ .

*Proof.* The proof of properties 1–3 is based on known properties of solutions to ODE's. We prove the generalized triangle inequality at the end of current subsection (see Corollary 2.2.14).  $\square$

**Theorem 2.2.7.** *Let  $X_j \in C^1$ . Fix  $u \in \mathbb{M}$ . If  $d_\infty(u, w) = C\varepsilon$ , then*

$$\widehat{X}_j^u(w) = \sum_{k: \deg X_k \leq \deg X_j} [\delta_{kj} + O(\varepsilon)] X_k + \sum_{k: \deg X_k > \deg X_j} o(\varepsilon^{\deg X_k - \deg X_j}) X_k(w),$$

$j = 1, \dots, N$ . All  $o(\cdot)$  are uniform in  $u$  belonging to some compact subset of  $U$ .

*Proof.* 1<sup>ST</sup> STEP. Applying the mapping  $\theta_u^{-1}$  to each vector field  $\widehat{X}_j^u$ ,  $j = 1, \dots, N$ , we deduce

$$D\theta_u^{-1} \widehat{X}_j^u(s) = \sum_{k=1}^N z_j^k(s) e_k,$$

where  $\{e_k\}_{k=1}^N$  is the collection of the vectors of the standard basis in  $\mathbb{R}^N$ , and by (2.1.5)

$$z_j^k(s) = \delta_{kj} + \sum_{|\mu|_h = \deg X_k - \deg X_j > 0} F_{\mu, e_j}^k(u) s^\mu.$$

Note that, here  $|s^\mu| = O(\varepsilon^{\deg X_k - \deg X_j})$ , since

$$d_\infty(0, s) = d_\infty(\theta_u^{-1}(u), s) = d_\infty^u(u, \theta_u(s)) = O(\varepsilon).$$

Then

$$\widehat{X}_j^u(\theta_u(s)) = \sum_{k=1}^N z_j^k(s) D\theta_u(s) e_k = \sum_{k=1}^N z_j^k(s) \left( X_k(\theta_u(s)) + \frac{1}{2} \left[ X_k, \sum_{l=1}^N s_l X_l \right] (\theta_u(s)) \right),$$

since  $D\theta_u(s) e_k = X_k(\theta_u(s)) + \frac{1}{2} \left[ X_k, \sum_{l=1}^N s_l X_l \right] (\theta_u(s))$ , where  $s = (s_1, \dots, s_N)$ .

To understand the latter, it is enough to consider the following equalities

$$\begin{aligned}
\theta_u(s + re_k) &= \exp\left(\sum_{l=1}^N s_l X_l + rX_k\right)(u) \\
&= \exp\left(\sum_{l=1}^N s_l X_l + rX_k\right) \circ \exp\left(-\sum_{l=1}^N s_l X_l\right) \circ \exp\left(\sum_{l=1}^N s_l X_l\right)(u) \\
&= \exp\left(rX_k + \frac{r}{2}\left[X_k, \sum_{l=1}^N s_l X_l\right] + o(r)\right)(\theta_u(s)),
\end{aligned}$$

and note (see justification of this calculation for  $C^1$ -vector fields in [2]) that

$$D\theta_u(s)e_k = \frac{\partial}{\partial r}\theta_u(s + re_k)\Big|_{r=0} = X_k(\theta_u(s)) + \frac{1}{2}\left[X_k, \sum_{l=1}^N s_l X_l\right](\theta_u(s)).$$

In view of the properties of the point  $s$ , we get  $|s_l| = O(\varepsilon^{\deg X_l})$ ,  $l = 1, \dots, N$ . Moreover, taking into account the definition of a Carnot manifold, we have

$$\left[X_k, \sum_{l=1}^N s_l X_l\right](\theta_u(s)) = \sum_{l=1}^N \sum_{m: \deg X_m \leq \deg X_k + \deg X_l} c_{klm}(\theta_u(s)) X_m(\theta_u(s)).$$

Consequently,

$$\begin{aligned}
\widehat{X}_j^u(\theta_u(s)) &= \sum_{k=1}^N z_j^k(s) X_k(\theta_u(s)) \\
&\quad + \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N \sum_{\deg X_m \leq \deg X_k + \deg X_l} z_j^k(s) s_l c_{klm}(\theta_u(s)) X_m(\theta_u(s)) \\
&= \sum_{k=1}^N \left[ z_j^k(s) + \frac{1}{2} \sum_{m, l: \deg X_k \leq \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlk}(\theta_u(s)) \right] X_k(\theta_u(s)).
\end{aligned}$$

where  $|z_j^m(s)| = O(\varepsilon^{\deg X_m - \deg X_j})$  and

$$|z_j^m(s) s_l| = O(\varepsilon^{\deg X_m + \deg X_l - \deg X_j}). \quad (2.2.1)$$



Represent the last sum as

$$\begin{aligned}
& \sum_{k: \deg X_k < \deg X_j} \left[ z_j^k(s) + \frac{1}{2} \sum_{m, l: \deg X_k \leq \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlk}(\theta_u(s)) \right] X_k(\theta_u(s)) \\
& + \sum_{k: \deg X_k = \deg X_j} \left[ z_j^k(s) + \frac{1}{2} \sum_{m, l: \deg X_j \leq \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlj}(\theta_u(s)) \right] X_j(\theta_u(s)) \\
& + \sum_{k: \deg X_k > \deg X_j} \left[ z_j^k(s) + \frac{1}{2} \sum_{m, l: \deg X_k \leq \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlk}(\theta_u(s)) \right] X_k(\theta_u(s)).
\end{aligned} \tag{2.2.2}$$

Note that, we have  $z_j^k(s) = 0$  if  $k < j$ . Next, if  $k < j$  and  $\deg X_k = \deg X_m + \deg X_l$ , we have  $m < j$  and  $z_j^m(s) = 0$ . Thus, for the first sum equals

$$\sum_{k: \deg X_k < \deg X_j} \left[ \frac{1}{2} \sum_{m, l: \deg X_k < \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlk}(\theta_u(s)) \right] X_k(\theta_u(s)).$$

Similarly, for the second sum we have  $z_j^k(s) = \delta_{kj}$ , and if  $\deg X_j = \deg X_m + \deg X_l$  then  $z_j^m(s) = 0$  since this relation implies  $m < j$ . Thus, we obtain

$$\sum_{k: \deg X_k = \deg X_j} \left[ \delta_{kj} + \frac{1}{2} \sum_{m, l: \deg X_j < \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlj}(\theta_u(s)) \right] X_j(\theta_u(s)).$$

In the third sum, the functions  $z_j^k(s)$  and  $z_j^m(s)$  can take any possible values.  
2<sup>ND</sup> STEP. Now, we calculate more exact estimates of (2.2.1).

- Let  $\deg X_k > \deg X_j$  and  $\deg X_k = \deg X_m + \deg X_l$ . From the above estimate we infer

$$|z_j^m(s) s_l| = O(\varepsilon^{\deg X_k - \deg X_j}).$$

Next, suppose that  $\deg X_k > \deg X_j$  and  $\deg X_k < \deg X_m + \deg X_l$ . Then all the situations  $\deg X_m > \deg X_j$ ,  $\deg X_m = \deg X_j$  and  $\deg X_m < \deg X_j$  are possible. Here we have

$$|z_j^m(s) s_l| = \begin{cases} \varepsilon O(\varepsilon^{\deg X_l}) \leq \varepsilon O(\varepsilon^{\deg X_k - \deg X_j}) & \text{if } \deg X_m > \deg X_j, \\ O(\varepsilon^{\deg X_l}) \leq \varepsilon O(\varepsilon^{\deg X_k - \deg X_j}) & \text{if } \deg X_m = \deg X_j, \\ 0 & \text{if } \deg X_m < \deg X_j. \end{cases}$$

- Let now  $\deg X_k = \deg X_j$  and  $\deg X_k < \deg X_m + \deg X_l$ . We again have to consider the situations  $\deg X_m > \deg X_j$ ,  $\deg X_m = \deg X_j$  and  $\deg X_m < \deg X_j$ . It follows

$$|z_j^m(s)s_l| = \begin{cases} \varepsilon O(\varepsilon^{\deg X_l}) \leq \varepsilon O(1) & \text{if } \deg X_m > \deg X_j, \\ O(\varepsilon^{\deg X_l}) \leq \varepsilon O(1) & \text{if } \deg X_m = \deg X_j, \\ 0 & \text{if } \deg X_m < \deg X_j. \end{cases}$$

- Finally, let  $\deg X_k < \deg X_j$  and  $\deg X_k < \deg X_m + \deg X_l$ . In three situations  $\deg X_m > \deg X_j$ ,  $\deg X_m = \deg X_j$  and  $\deg X_m < \deg X_j$ , we obtain the same result as in the previous case:

$$|z_j^m(s)s_l| = \begin{cases} \varepsilon O(\varepsilon^{\deg X_l}) \leq \varepsilon O(1) & \text{if } \deg X_m > \deg X_j, \\ O(\varepsilon^{\deg X_l}) \leq \varepsilon O(1) & \text{if } \deg X_m = \deg X_j, \\ 0 & \text{if } \deg X_m < \deg X_j. \end{cases}$$

Thus, in the first sum of (2.2.2), the coefficients at  $X_k$  equal  $O(\varepsilon)$ , and in the second sum the coefficient at  $X_j$  equals  $1 + O(\varepsilon)$ , and the coefficients at  $X_k$  for  $k \neq j$  equal  $O(\varepsilon)$ .

3<sup>RD</sup> STEP. Consider the last sum (where  $\deg X_k > \deg X_j$ ). Note that,

$$c_{mlk}(\theta_u(s)) = c_{mlk}(u) + o(1). \quad (2.2.3)$$

Then, taking into account (2.2.1) and the results of the 2<sup>nd</sup> step, we deduce

$$\begin{aligned} & \sum_{m,l:\deg X_k \leq \deg X_m + \deg X_l} z_j^m(s)s_l c_{mlk}(\theta_u(s)) \\ &= \sum_{m,l:\deg X_k = \deg X_m + \deg X_l} z_j^m(s)s_l c_{mlk}(\theta_u(s)) \\ &+ \sum_{m,l:\deg X_k < \deg X_m + \deg X_l} z_j^m(s)s_l c_{mlk}(\theta_u(s)) \\ &= \sum_{m,l:\deg X_k = \deg X_m + \deg X_l} z_j^m(s)s_l c_{mlk}(u) + o(1) \cdot \varepsilon^{\deg X_k - \deg X_j} \\ &\quad + \varepsilon \cdot O(\varepsilon^{\deg X_k - \deg X_j}) \\ &= \sum_{m,l:\deg X_k = \deg X_m + \deg X_l} z_j^m(s)s_l c_{mlk}(u) + o(1) \cdot \varepsilon^{\deg X_k - \deg X_j}. \quad (2.2.4) \end{aligned}$$

Consequently,

$$\begin{aligned} \widehat{X}_j^u(\theta_u(s)) &= \sum_{k: \deg X_k \leq \deg X_j} [\delta_{kj} + O(\varepsilon)] X_k \\ + \sum_{k: \deg X_k > \deg X_j} \left[ z_j^k(s) + \frac{1}{2} \sum_{m,l} z_j^m(s) s_l c_{mlk}(u) + o(\varepsilon^{\deg X_k - \deg X_j}) \right] X_k(\theta_u(s)), \end{aligned}$$

where  $m, l$  in the last sum are such that  $\deg X_k = \deg X_m + \deg X_l$ .

4<sup>TH</sup> STEP. It only remains to show that

$$z_j^k(s) + \frac{1}{2} \sum_{m,l: \deg X_k = \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlk}(u) = \delta_{kj}. \quad (2.2.5)$$

For obtaining this, consider the mapping  $\widehat{\theta}_u(x) = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(u) = \theta_u(x)$ , and apply the arguments of the 1<sup>st</sup> step with the following difference: it is known, that the vector fields  $\widehat{X}_i^u$ ,  $i = 1, \dots, N$ , are continuous, but they may not be differentiable, and formally, we cannot consider commutators of such vector fields. Therefore we modify previous arguments. For doing this, we consider the following representation of the identical mapping:

$$\widehat{\theta}_0(s) = \exp\left(\sum_{i=1}^N s_i D\widehat{\theta}_u^{-1}(\widehat{X}_i^u)\right)(0) = s,$$

and represent  $e_k = D\widehat{\theta}_0(s)(e_k)$  as before we represented  $D\theta_u(s)(e_k)$ . It is possible, since the vector fields  $D\widehat{\theta}_u^{-1}(\widehat{X}_i^u)$ ,  $i = 1, \dots, N$ , are smooth. Similarly to the 1<sup>st</sup> step, we infer

$$D\widehat{\theta}_0(s)(e_k) = D\widehat{\theta}_u^{-1}(\widehat{X}_k^u)(\widehat{\theta}_0(s)) + \frac{1}{2} \left[ D\widehat{\theta}_u^{-1}(\widehat{X}_k^u), \sum_{l=1}^N s_l D\widehat{\theta}_u^{-1}(\widehat{X}_l^u) \right] (\widehat{\theta}_0(s)).$$

Since  $\widehat{\theta}_0(s) = s$  and in view of properties of the vector fields  $D\widehat{\theta}_u^{-1}(\widehat{X}_i^u)$ ,  $i = 1, \dots, N$ , we deduce

$$\begin{aligned} e_k &= D\widehat{\theta}_u^{-1}(\widehat{X}_k^u)(s) + \frac{1}{2} \left[ D\widehat{\theta}_u^{-1}(\widehat{X}_k^u), \sum_{l=1}^N s_l D\widehat{\theta}_u^{-1}(\widehat{X}_l^u) \right] (s) \\ &= D\widehat{\theta}_u^{-1}(\widehat{X}_k^u)(s) + \frac{1}{2} \sum_{l=1}^N s_l \sum_{\deg X_m = \deg X_k + \deg X_l} c_{klm}(u) D\widehat{\theta}_u^{-1}(\widehat{X}_m^u)(s). \end{aligned}$$

It follows

$$D\widehat{\theta}_u(s)e_k = \widehat{X}_k^u(\theta_u(s)) + \frac{1}{2} \sum_{l=1}^N s_l \sum_{\deg X_m = \deg X_k + \deg X_l} c_{klm}(u) \widehat{X}_m^u(\theta_u(s)).$$

Applying further the arguments of the 1<sup>st</sup> step, we have

$$\widehat{X}_j^u(\theta_u(s)) = \sum_{k=1}^N \left[ z_j^k(s) + \frac{1}{2} \sum_{m,l: \deg X_k = \deg X_m + \deg X_l} z_j^m(s) s_l c_{mlk}(u) \right] \widehat{X}_k^u(\theta_u(s)),$$

and thus (2.2.5) is proved.

Taking into account the result of the 3<sup>rd</sup> step, we obtain

$$\widehat{X}_j^u(w) = \sum_{k: \deg X_k \leq \deg X_j} [\delta_{kj} + O(\varepsilon)] X_k + \sum_{k: \deg X_k > \deg X_j} o(\varepsilon^{\deg X_k - \deg X_j}) X_k(w),$$

$j = 1, \dots, N$ . The theorem follows.  $\square$

**Remark 2.2.8. 1.** If the vector fields  $X_i$ ,  $i = 1, \dots, N$ , belong to the class  $C^{1,\alpha}$ ,  $\alpha \in (0, 1]$ , then in (2.2.3) and, consequently, in (2.2.4), we obtain  $o(1) = O(\rho(u, \theta_u(s))^\alpha)$ . In this case, we have

$$\begin{aligned} \widehat{X}_j^u(w) = & \sum_{k: \deg X_k \leq \deg X_j} [\delta_{kj} + O(\varepsilon)] X_k \\ & + \sum_{k: \deg X_k > \deg X_j} \rho(u, \theta_u(s))^\alpha \cdot o(\varepsilon^{\deg X_k - \deg X_j}) X_k(w). \end{aligned}$$

**2.** If the derivatives of the basis vector fields are Hölder with respect to  $d_\infty$ , we obtain  $o(1) = O(d_\infty(u, \theta_u(s))^\alpha) = O(\varepsilon^\alpha)$ , and

$$\widehat{X}_j^u(w) = \sum_{k: \deg X_k \leq \deg X_j} [\delta_{kj} + O(\varepsilon)] X_k + \sum_{k: \deg X_k > \deg X_j} O(\varepsilon^{\deg X_k - \deg X_j + \alpha}) X_k(w).$$

**3.** If the derivatives of the basis vector fields are Hölder with respect to  $\mathfrak{d}$ , we have

$$\begin{aligned} \widehat{X}_j^u(w) = & \sum_{k: \deg X_k \leq \deg X_j} [\delta_{kj} + O(\varepsilon)] X_k \\ & + \sum_{k: \deg X_k > \deg X_j} \mathfrak{d}(u, \theta_u(s))^\alpha \cdot o(\varepsilon^{\deg X_k - \deg X_j}) X_k(w). \end{aligned}$$

**Corollary 2.2.9.** *For  $x \in \text{Box}(u, \varepsilon)$ , the coefficients  $\{a_{j,k}(x)\}_{j,k=1}^N$  from the equality*

$$X_j(x) = \sum_{k=1}^N a_{j,k}(x) \widehat{X}_k^u(x) \quad (2.2.6)$$

*enjoy the following property:*

$$a_{j,k}(x) = \begin{cases} O(\varepsilon) & \text{if } \deg X_j > \deg X_k, \\ \delta_{kj} + O(\varepsilon) & \text{if } \deg X_j = \deg X_k, \\ o(\varepsilon^{\deg X_k - \deg X_j}) & \text{if } \deg X_j < \deg X_k, \end{cases} \quad (2.2.7)$$

$j = 1, \dots, N$ . All “ $o$ ” are uniform in  $u$  belonging to some compact subset of  $U$ .

*Proof.* According to Theorem 2.2.7, the coefficients  $b_{j,k}(x)$  from the relation

$$\widehat{X}_j^u(x) = \sum_{k=1}^N b_{j,k}(x) X_k(x),$$

$j = 1, \dots, N$ , have the same properties. Put  $A(x) = (a_{j,k}(x))_{j,k=1}^N$  and  $B(x) = (b_{j,k}(x))_{j,k=1}^N$ . Then  $A(x) = B(x)^{-1}$ .

We use the well-known formula of calculation of the entries of the inverse matrix to estimate all  $a_{j,k}(x)$ ,  $j, k = 1, \dots, N$ . We estimate the value  $|a_{j,k}(x)| = \frac{|\det B_{j,k}(x)|}{|\det B(x)|}$ , where the  $(N-1) \times (N-1)$ -matrix  $B_{j,k}$  is constructed from the matrix  $B(x)$  by deleting its  $j$ th column and  $k$ th line.

It is easy to see that  $|\det B(x)| = 1 + O(\varepsilon)$ , where  $O(\varepsilon)$  is uniform for  $x$  belonging to some compact neighborhood  $U \subset \mathbb{M}$ .

Next, we estimate  $|\det B_{j,k}(x)|$ . Obviously,  $|\det B_{j,j}(x)| = 1 + O(\varepsilon)$ , where  $O(\varepsilon)$  is uniform for  $x$  belonging to some compact neighborhood  $U \subset \mathbb{M}$ ,  $j = 1, \dots, N$ .

Let now  $k > j$ . By construction, the diagonal elements with numbers  $(i, i)$ ,  $j \leq i < k$ , equal  $o(\varepsilon^{\deg X_{i+1} - \deg X_i})$ , and the elements under these ones equal  $1 + O(\varepsilon)$ . Note that,  $\det B_{j,k}(x)$  up to a multiple  $(1 + O(\varepsilon))$  equals the product of determinants of the following three matrices: the first  $P(x) = p_{i,l}(x)$  is a  $(j-1) \times (j-1)$ -matrix with  $p_{i,l}(x) = b_{i,l}(x)$ , the second  $Q(x) = q_{i,l}(x)$  is a  $(k-j) \times (k-j)$ -matrix with  $q_{i,l}(x) = b_{i+j-1, l+j}(x)$ , and the third  $R(x) = r_{i,l}(x)$  with  $r_{i,l}(x) = b_{i+k-1, l+k-1}(x)$ .

For the matrices  $P(x)$  and  $R(x)$  we have  $|\det P(x)| = 1 + O(\varepsilon)$  and  $|\det R(x)| = 1 + O(\varepsilon)$ . By construction,  $q_{i,i}(x) = o(\varepsilon^{\deg X_{i+1} - \deg X_i})$  and  $q_{i+1,i}(x) = 1 + O(\varepsilon)$ . We have that the product of the diagonal elements of

$Q(x)$  equals

$$\prod_{i=j}^{k-1} o(\varepsilon^{\deg X_{i+1} - \deg X_i}) = o(\varepsilon^{\deg X_k - \deg X_j}).$$

It is easy to see that, for all other summands constituting  $\det Q(x)$ , we have the same estimate.

Similarly, we show that for  $k < j$  we have  $|\det B_{jk}(x)| = O(\varepsilon)$ . Here  $O(\varepsilon)$  is uniform for  $x$  belonging to some compact neighborhood  $U \subset \mathbb{M}$ . The lemma follows.  $\square$

**Remark 2.2.10.** Similarly to Remark 2.2.8:

- if  $X_i \in C^{1,\alpha}$  then

$$a_{j,k}(x) = \begin{cases} O(\varepsilon) & \text{if } \deg X_j > \deg X_k, \\ \delta_{kj} + O(\varepsilon) & \text{if } \deg X_j = \deg X_k, \\ \rho(u, x)^\alpha \cdot o(\varepsilon^{\deg X_k - \deg X_j}) & \text{if } \deg X_j < \deg X_k, \end{cases}$$

- if the derivatives of the basis vector fields are Hölder with respect to  $d_\infty$  then

$$a_{j,k}(x) = \begin{cases} O(\varepsilon) & \text{if } \deg X_j > \deg X_k, \\ \delta_{kj} + O(\varepsilon) & \text{if } \deg X_j = \deg X_k, \\ O(\varepsilon^{\deg X_k - \deg X_j + \alpha}) & \text{if } \deg X_j < \deg X_k, \end{cases}$$

- if the derivatives of the basis vector fields are Hölder with respect to  $\mathfrak{d}$  then

$$a_{j,k}(x) = \begin{cases} O(\varepsilon) & \text{if } \deg X_j > \deg X_k, \\ \delta_{kj} + O(\varepsilon) & \text{if } \deg X_j = \deg X_k, \\ \mathfrak{d}(u, x)^\alpha \cdot O(\varepsilon^{\deg X_k - \deg X_j}) & \text{if } \deg X_j < \deg X_k, \end{cases}$$

$j = 1, \dots, N$ .

Corollary 2.2.9 imply instantly Gromov's Theorem on the nilpotentization of vector fields [68].

**Corollary 2.2.11** (Gromov's Theorem [68]). *Let  $X_j \in C^1$ . On  $\text{Box}(g, \varepsilon r_g)$ , consider the vector fields  $\{\varepsilon X_i\} = \{\varepsilon^{\deg X_i} X_i\}$ ,  $i = 1, \dots, N$ . Then the uniform convergence*

$$X_i^\varepsilon = (\Delta_{\varepsilon^{-1}}^g)_* \varepsilon X_i \rightarrow \widehat{X}_i^g \quad \text{as } \varepsilon \rightarrow 0, \quad i = 1, \dots, N,$$

*holds at the points of the box  $\text{Box}(g, r_g)$  and this convergence is uniform in  $g$  belonging to some compact set.*

*Proof.* Really, by (2.2.6), (2.2.7) and in view of Corollary 2.2.9 and Property 2.1.20, we infer

$$\begin{aligned}
X_i^\varepsilon(x) &= ((\Delta_{\varepsilon^{-1}}^g)_*^\varepsilon X_i)(x) = \varepsilon^{\deg X_i} \sum_{k=1}^N a_{i,k}(\Delta_\varepsilon^g(x)) ((\Delta_{\varepsilon^{-1}}^g)_*^\varepsilon \widehat{X}_k^g)(x) \\
&= \sum_{k=1}^N \varepsilon^{\deg X_i - \deg X_k} a_{i,k}(\Delta_\varepsilon^g(x)) \widehat{X}_k^g(x) \\
&= \sum_{k: \deg X_k \leq \deg X_i} \varepsilon^{\deg X_i - \deg X_k} (\delta_{ik} + O(\varepsilon)) \widehat{X}_k^g(x) + \sum_{k: \deg X_k > \deg X_i} o(1) \widehat{X}_k^g(x)
\end{aligned}$$

as  $\varepsilon \rightarrow 0$ . It follows the uniform convergence  $X_i^\varepsilon = (\Delta_{\varepsilon^{-1}}^g)_*^\varepsilon X_i \rightarrow \widehat{X}_i^g$  as  $\varepsilon \rightarrow 0$ ,  $i = 1, \dots, N$ , at the points of the box  $\text{Box}(g, r_g)$  and this convergence is uniform in  $g$  belonging to some compact set.  $\square$

**Remark 2.2.12.** For  $C^\infty$ -vector fields, the above corollary is formulated in [101, 119] in another way:  $\widehat{X}_i^g$  is an homogeneous part of  $X_i$ ,  $i = 1, \dots, N$ . This statement implies Corollary 2.2.11. It is shown in [67] that, applying similar arguments, the smoothness of vector fields can be reduced to be  $2M + 1$ .

Estimates (2.2.7) were written in the proof of [130, Theorem 3.1] as a consequence of the Gromov's Theorem which can be proved by method of [119] under an additional smoothness of vector fields:  $X_j \in C^{2M - \deg X_j}$ . Corollary 2.2.11 shows that estimates (2.2.7) are not only necessary but also sufficient for the validity of the Gromov's Theorem. In our paper estimates (2.2.7) are obtained under minimal assumption on the smoothness of vector fields:  $X_j \in C^1$ ,  $j = 1, \dots, N$ . Thus, taking into account the footnote in [130, p. 253], all results of papers [127, 128, 129, 130, 132] are valid under the same assumptions on the smoothness of basis vector fields.

Recall that Gromov [68, p. 130] has formulated the theorem under assumption  $X_j \in C^1$ . Valeriĭ Berestovskii sent us the following example confirming that arguments of Gromov's proof have to be corrected.

**Example.** Let  $X = \frac{\partial}{\partial x}$ ,  $Y = xy \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$ . Then  $Z := [X, Y] = y \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$ ,  $[X, Z] = 0$ ,  $[Y, Z] = \frac{\partial}{\partial x} - y(y \frac{\partial}{\partial x} + \frac{\partial}{\partial z}) = (1 - y^2) \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}$ . One can easily see that  $X, Y, Z$  constitutes a global frame of smooth vector fields over the ring of smooth functions in  $\mathbb{R}^3$ . Also for corresponding one-parameter subgroups  $X(x)$ ,  $Y(y)$ ,  $Z(z)$ , we have  $(X(x) \circ Y(y) \circ Z(z))(0, 0, 0) = (x, y, z)$ . Under this  $X = \frac{\partial}{\partial x}$  on  $\mathbb{R}^3$ ,  $Y = \frac{\partial}{\partial y}$  on  $x = 0$ ,  $Z = \frac{\partial}{\partial z}$  on  $z$ -line (even on  $y = 0$ ). On the other hand,  $\frac{\partial}{\partial y} Z = X \neq [Y, Z]$  (see above) on  $x = 0$ . This contradicts to the Gromov's statement that (A) of [68, p. 131] implies (B) of [68, p. 132] in general case.

**Corollary 2.2.13** (Estimate of the Diameter of a Box). *In a compact neighborhood  $U \subset \mathbb{M}$ , for each point  $u \in U$  and each  $\varepsilon > 0$  small enough, we have  $\text{diam}(\text{Box}(u, \varepsilon)) \leq L\varepsilon$ , where  $L$  depends only on  $U$ .*

*Proof.* Assume the contrary: there exist sequences  $\{\varepsilon_k\}_{k \in \mathbb{N}}$ ,  $\{u_k\}_{k \in \mathbb{N}}$ ,  $\{v_k\}_{k \in \mathbb{N}}$  and  $\{w_k\}_{k \in \mathbb{N}}$  such that  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ ,  $d_\infty(u_k, v_k) = \varepsilon_k$  and  $d_\infty(u_k, w_k) \leq \varepsilon_k$  but  $d_\infty(v_k, w_k) > k\varepsilon_k$ . Since  $U \subset \mathbb{M}$  is compact, we may assume without loss of generality that  $u_k \rightarrow u_0$  as  $k \rightarrow \infty$ . Then  $v_k \rightarrow u_0$  and  $w_k \rightarrow u_0$  as  $k \rightarrow \infty$ .

Assume without loss of generality that  $\varepsilon^{\deg X_i} D\Delta_{\varepsilon^{-1}}^{u_k} X_i(x) \rightarrow \widehat{X}_i^{u_k}(x)$  as  $\varepsilon \rightarrow 0$  for  $x \in \text{Box}(u_0, Kr_0)$  uniformly in  $u_k$ ,  $i = 1, \dots, N$ , where  $K = \max\{5, 5c^4\}$ ,  $c$  is such that  $d_\infty^{u_k}(v, w) \leq c(d_\infty^{u_k}(u, v) + d_\infty^{u_k}(u, w))$  for all  $k \in \mathbb{N}$  big enough, and  $k \in \mathbb{N}$  is big enough (see Corollary 2.2.11). Note that,  $c < \infty$  since  $c(u_k)$  continuously depends on values of  $\{F_{\mu, \beta}^j(u_k)\}_{j, \mu, \beta}$ , consequently, it depends continuously on  $u_k$ . Moreover, the choice of  $K$  implies the following:

1. For  $k$  big enough, we have that an integral curve of a vector field with constant coefficients connecting  $\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(w_k)$  and  $\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(v_k)$  in the local Carnot group  $\mathcal{G}^{u_k} \mathbb{M}$  lies in  $\text{Box}(u_k, Kr_0)$ ;
2. We may choose  $k$  by the following way:  $d_\infty(u_0, u_k) < r_0$  and the Riemannian distance between the integral curves corresponding to the collections  $\{\widehat{X}_i^{u_k}\}_{i=1}^N$  and  $\{(r_0^{-1} \varepsilon_k)^{\deg X_i} D\Delta_{r_0 \varepsilon_k^{-1}}^{u_k} X_i\}_{i=1}^N$  (with constant coefficients) that connect points  $\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(w_k)$  and  $\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(v_k)$ , is less than  $r_0$ .

Fix  $k \in \mathbb{N}$ . Then  $v_k = \exp\left(\sum_{i=1}^N \xi_i \varepsilon_k^{\deg X_i} X_i\right)(u_k)$ ,  $w_k = \exp\left(\sum_{i=1}^N \eta_i \varepsilon_k^{\deg X_i} X_i\right)(u_k)$ , and  $w_k = \exp\left(\sum_{i=1}^N \zeta_i(\varepsilon_k) \varepsilon_k^{\deg X_i} X_i\right)(v_k)$ . Apply the mapping  $\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}$  to  $v_k$  and  $w_k$ . We have

$$\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(w_k) = \exp\left(\sum_{i=1}^N \zeta_i(\varepsilon_k) \varepsilon_k^{\deg X_i} D\Delta_{r_0 \varepsilon_k^{-1}}^{u_k} X_i\right)\left(\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(v_k)\right).$$

Note that,  $d_\infty(u_k, \Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(v_k)) = r_0$  and  $d_\infty(u_k, \Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(w_k)) \leq r_0$ . In view of Corollary 2.2.11, the vector fields  $(r_0^{-1} \varepsilon_k)^{\deg X_i} D\Delta_{r_0 \varepsilon_k^{-1}}^{u_k} X_i(x) = \widehat{X}_i^{u_k}(x) + o(1)$ ,  $i = 1, \dots, N$ , where  $o(1)$  is uniform in  $x$  and in  $u_k$ . Consequently, since  $\dim \text{span}\{\widehat{X}_i^{u_k}(x)\}_{i=1}^N = N$  at each  $x \in \text{Box}(u_0, r_0)$ , the Riemannian distance between  $\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(w_k)$  and  $\Delta_{r_0 \varepsilon_k^{-1}}^{u_k}(v_k)$  is bounded from above for all  $k \in \mathbb{N}$  big enough. Therefore, the coefficients  $\zeta_i(\varepsilon_k)$ ,  $i = 1, \dots, N$ , are bounded from



above for all  $k \in \mathbb{N}$  big enough. The assumption  $d_\infty(v_k, w_k) > k\varepsilon_k$  contradicts this conclusion.

Thus there exists a constant  $L = L(U)$  such that  $\text{diam}(\text{Box}(u, \varepsilon)) \leq L\varepsilon$  for  $u \in U$ . The statement follows.  $\square$

From the previous statement we come immediately to the following

**Corollary 2.2.14** (Triangle inequality). *The quasimetric  $d_\infty(x, y)$  meets locally the generalized triangle inequality (see Property 2.2.6).*

**Corollary 2.2.15** (Decomposition of the basis vector fields). *Fix a point  $\theta_u(s) \in \text{Box}(u, O(\varepsilon))$ . Remarks 2.2.8 and 2.2.10 imply the following decomposition of  $D\theta_u^{-1}X_i$ ,  $i = 1, \dots, N$ :*

$$[D\theta_u^{-1}X_i(s)]_j = [(\widehat{X}_i^u)'(s)]_j + \sum_{k=1}^N (a_{i,k}(\theta_u(s)) - \delta_{ik})[(\widehat{X}_k^u)'(s)]_j.$$

If  $d_\infty(u, \theta_u(s)) = O(\varepsilon)$ , we have

$$\begin{aligned} [D\theta_u^{-1}X_i(s)]_j &= z_i^j(u, s) + \sum_{k: \deg X_k \leq \deg X_i} O(\varepsilon) z_k^j(u, s) \\ &\quad + \sum_{k: \deg X_k > \deg X_i} a_{i,k}(\theta_u(s)) z_k^j(u, s). \end{aligned}$$

If  $\deg X_j \leq \deg X_i$  then  $[D\theta_u^{-1}X_i(s)]_j = \delta_{ij} + O(\varepsilon)$ . For  $\deg X_j > \deg X_i$  we have:

- If the basis vector fields are  $C^1$ -smooth then we deduce  $[D\theta_u^{-1}X_i(s)]_j = z_i^j(u, s) + O(\varepsilon^{\deg X_j - \deg X_i + 1}) + o(1) \cdot \varepsilon^{\deg X_j - \deg X_i}$ , and therefore

$$[D\theta_u^{-1}X_i(s)]_j = z_i^j(u, s) + o(\varepsilon^{\deg X_j - \deg X_i}).$$

- If the derivatives of the basis vector fields are  $H^\alpha$ -continuous with respect to  $\mathfrak{d}$ , then if  $\deg X_j > \deg X_i$  we have

$$[D\theta_u^{-1}X_i(s)]_j = z_i^j(u, s) + \mathfrak{d}(u, \theta_u(s))^\alpha \cdot O(\varepsilon^{\deg X_j - \deg X_i}).$$

In particular, for  $\alpha = 1$  and  $\mathfrak{d} = d_\infty$  or  $\mathfrak{d} = d_\infty^z$ , where  $d_\infty(u, z) = O(\varepsilon)$ , we have

$$[D\theta_u^{-1}X_i(s)]_j = z_i^j(u, s) + O(\varepsilon^{\deg X_j - \deg X_i + 1}).$$

## 2.3 Comparison of Geometries of Tangent Cones

The goal of Subsections 2.3, 2.4 and 2.6 is to compare the geometries of two local Carnot groups. The main result of Section 2 is the following

**Theorem 2.3.1.** *Let  $u, u' \in U$  be such that  $d_\infty(u, u') = C\varepsilon$ . For a fixed  $Q \in \mathbb{N}$ , consider points  $w_0$ ,  $d_\infty(u, w_0) = C\varepsilon$ , and*

$$w_j^\varepsilon = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(w_{j-1}^\varepsilon), \quad w_j^{\varepsilon'} = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\right)(w_{j-1}^{\varepsilon'}),$$

$w_0^{\varepsilon'} = w_0^\varepsilon = w'_0 = w_0$ ,  $j = 1, \dots, Q$ . (Here  $Q \in \mathbb{N}$  is such that all these points belong to the neighborhood  $U \subset \mathbb{M}$ , for all  $\varepsilon > 0$ .) Then for  $\alpha > 0$ ,

$$\max\{d_\infty^u(w_Q^\varepsilon, w_Q^{\varepsilon'}), d_\infty^{u'}(w_Q^\varepsilon, w_Q^{\varepsilon'})\} = \varepsilon \cdot [\Theta(C, \mathcal{C}, Q, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta})] \rho(u, u')^{\frac{\alpha}{M}}. \quad (2.3.1)$$

In the case of  $\alpha = 0$ , we have

$$\max\{d_\infty^u(w_Q^\varepsilon, w_Q^{\varepsilon'}), d_\infty^{u'}(w_Q^\varepsilon, w_Q^{\varepsilon'})\} = \varepsilon \cdot [\Theta(C, \mathcal{C}, Q, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta})][\omega(\rho(u, u'))]^{\frac{1}{M}}$$

where  $\omega \rightarrow 0$  is a modulus of continuity. (Here  $\Theta$  is uniform in  $u, u', w_0 \in U$  and  $\{w_{i,j}\}$ ,  $i = 1, \dots, N$ ,  $j = 1, \dots, Q$ , belonging to some compact neighborhood of 0, and it depends on  $Q$  and  $\{F_{\mu,\beta}^j\}_{j,\mu,\beta}$ .)

**Remark 2.3.2.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , then we have  $\mathfrak{d}(u, u')^{\frac{\alpha}{M}}$  instead of  $\rho(u, u')^{\frac{\alpha}{M}}$  in (2.3.1).

In the current subsection we prove the "base" of the main result, i. e., we obtain it for  $Q = 1$  and  $\varepsilon = 1$ . The full proof is written in Subsection 2.6.

Fix points  $u, u' \in U$ , where  $U$  is such that Assumption 2.1.4 holds. Recall that the collections of vector fields  $\{\widehat{X}_i^u\}_{i=1}^N$  and  $\{\widehat{X}_i^{u'}\}_{i=1}^N$  are frames in  $\mathcal{G}^u\mathbb{M}$  and in  $\mathcal{G}^{u'}\mathbb{M}$  respectively.

**Definition 2.3.3.** By  $\widehat{X}^p(q)$ , we denote the matrix, such that its  $i$ th column consists of the coordinates of the vector  $\widehat{X}_i^p(q)$ ,  $i = 1, \dots, N$ ,  $p \in \mathbb{M}$ ,  $q \in \mathcal{G}^p\mathbb{M}$ , in the frame  $\{\widehat{X}_j\}_{j=1}^N$ .

**Lemma 2.3.4.** *Suppose that Assumption 2.1.4 holds. Let  $\Xi(u, u', q)$ ,  $q \in \mathbb{M}$ , be the matrix such that*

$$\widehat{X}^{u'}(q) = \widehat{X}^u(q) \Xi(u, u', q). \quad (2.3.2)$$

*Then the entries of  $\Xi(u, u', q)$  are (locally)  $H^\alpha$ -continuous in  $u$  and  $u'$ .*

*Proof.* The proof of this statement follows from Theorem 2.1.12. Indeed, it implies that the vector fields  $\{\widehat{X}_i^u\}_{i=1}^N$  are locally  $H^\alpha$ -continuous in  $u$ . Since we prove a local property, and  $\mathbb{M}$  is a Riemannian manifold, then, instead of  $\mathbb{M}$ , we may consider without loss of generality some neighborhood  $U \subset \mathbb{R}^N$  containing  $u$  and  $u'$ . Then it is easy to see that the entries of the matrices  $\widehat{X}^u$  and  $\widehat{X}^{u'}$  are (locally)  $H^\alpha$ -continuous on  $U \times U$ . Since both matrices are non-degenerate in  $U \subset \mathbb{M}$ , we have that  $\Xi(u, u', q) = \widehat{X}^u(q)^{-1} \widehat{X}^{u'}(q)$  is also non-degenerate, and its entries  $\Xi_{ij}(u, u', q)$  belong locally to  $C^\alpha(U \times U)$ ,  $i, j = 1, \dots, N$ .  $\square$

**Remark 2.3.5.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , then the entries of  $\Xi$  are also locally Hölder with respect to  $\mathfrak{d}$  (see Remark 2.1.17).

**Remark 2.3.6.** Suppose that Assumption 2.1.4 holds. Since  $\Xi(u, u', q)$  equals the unit matrix if  $u = u'$  then  $\Xi_{ij} = \delta_{ij} + \Theta \rho(u, u')^\alpha$  where  $\Theta = \Theta(u, u', q)$  is a bounded measurable function:  $|\Theta| \leq C$ , and the constant  $C \geq 0$  depends only on the neighborhood  $U \subset \mathbb{M}$ .

*Proof.* Note that  $\Xi(u, u, q)$  equals the unit matrix. Then the  $\alpha$ -Hölder continuity of all vector fields implies  $|\Xi_{ij}(u, u', q) - \delta_{ij}| \leq C(\rho(u, u')^\alpha)$ , where

$$C = \sup_{u, u', q \in U} \frac{|\Xi_{ij}(u, u', q) - \delta_{ij}|}{\rho(u, u')^\alpha} < \infty$$

depends only on the neighborhood  $U \subset \mathbb{M}$ .  $\square$

**Remark 2.3.7.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , then  $\Xi_{ij}(u, u', q) = \delta_{ij} + \Theta \mathfrak{d}(u, u')^\alpha$ .

**Notation 2.3.8.** Throughout the paper, by the symbol  $\Theta$ , we denote some bounded function absolute values of which do not exceed some  $0 \leq C < \infty$ , where  $C$  depends only on the neighborhood where  $\Theta$  is defined (i. e., it does not depend on *points* of this neighborhood).

**Theorem 2.3.9.** *Let*

$$w = \exp\left(\sum_{i=1}^N w_i \widehat{X}_i^u\right)(v), \quad w' = \exp\left(\sum_{i=1}^N w_i \widehat{X}_i^{u'}\right)(v).$$

*Then, for  $\alpha > 0$ , we have*

$$\max\{d_\infty^u(w, w'), d_\infty^{u'}(w, w')\} = \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}, \quad (2.3.3)$$

*where  $u, u', v \in U$ ,  $\{w_i\}_{i=1}^N \in U(0) \subset \mathbb{R}^N$ .*

*In the case of  $\alpha = 0$ ,*

$$\max\{d_\infty^u(w, w'), d_\infty^{u'}(w, w')\} = \Theta[\omega_\Xi(\rho(u, u')) \rho(v, w)]^{\frac{1}{M}}.$$

**Remark 2.3.10.** Here (see Notation 2.3.8), the value

$$\sup |\Theta(u, u', v, \{w_i\}_{i=1}^N)| < \infty$$

depends only on  $U \subset \mathbb{M}$  and  $U(0) \subset \mathbb{R}^N$ .

*Proof of Theorem 2.3.9.* 1<sup>ST</sup> STEP. Fix  $q \in \mathbb{M}$ . Notice that both collections of vectors  $\{\widehat{X}_i^u(q)\}_{i=1}^N$  and  $\{\widehat{X}_i^{u'}(q)\}_{i=1}^N$  are frames of  $T_q\mathbb{M}$ . Consequently, there exists the transition  $(N \times N)$ -matrix  $\Xi(u, u', q) = (\Xi(u, u', q))_{i,k}$  such that

$$\widehat{X}_i^{u'}(q) = \sum_{k=1}^N (\Xi(u, u', q))_{i,k} \widehat{X}_k^u(q). \quad (2.3.4)$$

Remark 2.3.6 implies that

$$\Xi(u, u', q)_{i,j} = \begin{cases} 1 + \Theta_{i,j}\rho(u, u')^\alpha & \text{if } i = j, \\ \Theta_{i,j}\rho(u, u')^\alpha & \text{if } i \neq j. \end{cases} \quad (2.3.5)$$

Thus  $\widehat{X}_i^{u'}(q) = \widehat{X}_i^u(q) + [\Xi(u, u', q) - I]\widehat{X}_i^u(q)$  where  $||[\Xi(u, u', q) - I]|_{k,j} = \Theta_{k,j}\rho(u, u')^\alpha$  for all  $k, j = 1, \dots, N$ .

2<sup>ND</sup> STEP. Consider the integral line  $\gamma(t)$  of the vector field  $\sum_{i=1}^N w_i \widehat{X}_i^{u'}$  starting at  $v$  with the endpoint  $w'$ . Rewrite the tangent vector to  $\gamma(t)$  in the frame  $\{\widehat{X}_i^u\}_{i=1}^N$  as  $\dot{\gamma}(t) = \sum_{i=1}^N w_i^u(\gamma(t)) \widehat{X}_i^u(\gamma(t))$ . From (2.3.4) it follows that

$$w_i^u(q) = \sum_{k=1}^N w_k (\Xi(u, u', q))_{k,i}.$$

From (2.3.5) we can estimate the coefficient  $w_i^u$  at  $\widehat{X}_i^u$ :

$$w_i^u = w_i + \sum_{k=1}^N [w_k \Theta_{k,i} \rho(u, u')^\alpha], \quad i = 1, \dots, N. \quad (2.3.6)$$

3<sup>RD</sup> STEP. Next, we estimate the Riemannian distance between  $w$  and  $w'$ . By  $\kappa(t)$  denote the integral line of the vector field  $\sum_{i=1}^N w_i \widehat{X}_i^u$  connecting  $v$  and  $w$ , i. e., a line such that  $\kappa(0) = v$  and

$$\dot{\kappa}(t) = \sum_{i=1}^N w_i \widehat{X}_i^u(\kappa(t)).$$

By means of the mapping  $\theta_u^{-1}$  we transport  $\kappa(t)$  and  $\gamma(t)$  to  $\mathbb{R}^N$ . Let  $\kappa_u(t) = \theta_u^{-1}(\kappa(t))$  and  $\gamma_u(t) = \theta_u^{-1}(\gamma(t))$ . Then

$$\dot{\kappa}_u(t) = (\theta_u^{-1})_*(\kappa(t))\dot{\kappa}(t) = \sum_{i=1}^N w_i(\widehat{X}_i^u)'(\kappa_u(t))$$

and similarly

$$\dot{\gamma}_u(t) = \sum_{i=1}^N w_i(\theta_u^{-1})_*\widehat{X}_i^{u'} = \sum_{i=1}^N w_i^u(t)(\widehat{X}_i^u)'(\gamma_u(t))$$

since  $(\theta_u^{-1})_*\widehat{X}_i^{u'}(q) = \sum_{k=1}^N (\Xi(u, u', q))_{i,k}(\widehat{X}_i^u)'(q)$  (see (2.3.2)). Using formula (2.1.5) rewrite the tangent vectors in Cartesian coordinates:

$$\dot{\kappa}_u(t) = \sum_{i=1}^N w_i \sum_{j=1}^N z_i^j(u, \kappa_u(t)) \frac{\partial}{\partial x_j} = \sum_{j=1}^N W_j(u, \kappa_u(t)) \frac{\partial}{\partial x_j}$$

where

$$W_j(u, \kappa_u(t)) = \sum_{i=1}^N w_i z_i^j(u, \kappa_u(t)) = w_j + \sum_{i=1}^{j-1} w_i z_i^j(u, \kappa_u(t)).$$

Similarly

$$\dot{\gamma}_u(t) = \sum_{j=1}^N W_j^u(u, \gamma_u(t)) \frac{\partial}{\partial x_j}$$

where

$$W_j(u, \gamma_u(t)) = w_j^u(t) + \sum_{i=1}^{j-1} w_i^u(t) z_i^j(u, \gamma_u(t)).$$

Now we estimate the length of the curve  $\lambda_u(t) = \gamma_u(t) - \kappa_u(t) + \theta_u^{-1}(w)$  with endpoints  $\theta_u^{-1}(w)$  and  $\theta_u^{-1}(w')$ . The tangent vector to  $\lambda_u(t)$  equals

$$\begin{aligned} \dot{\lambda}_u(t) &= \dot{\gamma}_u(t) - \dot{\kappa}_u(t) = \sum_{j=1}^N [W_j^u(u, \gamma_u(t)) - W_j(u, \kappa_u(t))] \frac{\partial}{\partial x_j} \\ &= \sum_{j=1}^N \left[ (w_j^u(t) - w_j) + \sum_{i < j} w_i (z_i^j(u, \gamma_u(t)) - z_i^j(u, \kappa_u(t))) \right] \\ &\quad + \sum_{i,j=1}^N (w_i^u(t) - w_i) z_i^j(u, \gamma_u(t)). \quad (2.3.7) \end{aligned}$$

Notice that for the last sum we have

$$\sum_{i,j=1}^N (w_i^u(t) - w_i) z_i^j(u, \gamma_u(t)) = \Theta \rho(u, u')^\alpha \rho(v, w)$$

since  $w_i^u(t) = w_i + \Theta \rho(u, u')^\alpha \rho(v, w)$  by (2.3.6). By properties of  $z_i^j$ ,

$$z_i^j(u, \gamma_u(t)) - z_i^j(u, \kappa_u(t)) = \Theta \left[ \sum_{|\mu|=1} F_{\mu, e_i}^j (\gamma_u^\mu(t) - \kappa_u^\mu(t)) \right].$$

Notice that

$$|\gamma_u(t) - \kappa_u(t)| \leq \int_0^t |\dot{\gamma}_u(\tau) - \dot{\kappa}_u(\tau)| d\tau.$$

Consequently

$$\max_t |\gamma_u(t) - \kappa_u(t)| \leq \max_t |\dot{\gamma}_u(t) - \dot{\kappa}_u(t)| = \max_t |\dot{\lambda}_u(t)|.$$

Applying these estimates to (2.3.7) we obtain

$$\max_t |\dot{\lambda}_u(t)| = \Theta \rho(u, u')^\alpha \rho(v, w) + \Theta \rho(v, w) \max_t |\dot{\lambda}_u(t)|.$$

From here it follows

$$\max_t |\dot{\lambda}_u(t)| = \frac{\Theta \rho(u, u')^\alpha \rho(v, w)}{1 - \Theta \rho(v, w)} \leq \Theta \rho(u, u')^\alpha \rho(v, w)$$

if  $\Theta \rho(v, w) \leq \frac{1}{2}$ . Thus

$$\rho(\theta_u^{-1}(w), \theta_u^{-1}(w')) \leq \int_0^1 |\dot{\lambda}_u(t)| dt \leq \max_t |\dot{\lambda}_u(t)| = \Theta \rho(u, u')^\alpha \rho(v, w),$$

and  $\rho(w, w') \leq \Theta \rho(u, u')^\alpha \rho(v, w)$ .

4<sup>TH</sup> STEP. By the inequality  $d_\infty^u(p, q) \leq C \rho(p, q)^{\frac{1}{M}}$ , we obtain the estimate of  $d_\infty^u(w, w')$ :

$$d_\infty^u(w, w') = \Theta [\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}$$

in some compact neighborhood of  $g$ . The same estimate is true for  $d_\infty^{u'}(w, w')$ . The theorem follows.  $\square$

**Remark 2.3.11.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , then we have  $\mathfrak{d}(u, u')^\alpha$  instead of  $\rho(u, u')^\alpha$  in (2.3.3) (the proof is similar, see Remarks 2.1.17 and 2.3.7).

## 2.4 Comparison of Local Geometries of Tangent Cones

Consider points

$$w_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v) \text{ and } w'_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\right)(v).$$

**Theorem 2.4.1.** *Suppose that  $d_\infty(u, u') = C\varepsilon$  and  $d_\infty(u, v) = \mathcal{C}\varepsilon$  for some  $C, \mathcal{C} < \infty$ . Then, for  $\alpha > 0$ , we have*

$$\max\{d_\infty^u(w_\varepsilon, w'_\varepsilon), d_\infty^{u'}(w_\varepsilon, w'_\varepsilon)\} = \varepsilon[\Theta(C, \mathcal{C})]\rho(u, u')^{\frac{\alpha}{M}}. \quad (2.4.1)$$

In the case of  $\alpha = 0$ , we have

$$\begin{aligned} \max\{d_\infty^u(w_\varepsilon, w'_\varepsilon), d_\infty^{u'}(w_\varepsilon, w'_\varepsilon)\} \\ = \varepsilon[\Theta(C, \mathcal{C})] \max\{\omega_\Xi(\rho(u, u')), \omega_{\Delta_{\varepsilon^{-1}, v}^u \circ \Delta_{\varepsilon, v}^{u'}}(\rho(u, u'))\}^{\frac{\alpha}{M}}, \end{aligned}$$

where  $\Delta_{\varepsilon^{-1}, v}^u$  is defined below in (2.4.4) and (2.4.5). (Here  $\Theta$  is uniform in  $u, u', v \in U \subset \mathbb{M}$ , and in  $\{w_i\}_{i=1}^N$  belonging to some compact neighborhood of 0 (see Notation 2.3.8).)

**Remark 2.4.2.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally  $\alpha$ -Hölder with respect to  $\mathfrak{d}$  (instead of  $\rho$ ), then we have  $\mathfrak{d}(u, u')^{\frac{\alpha}{M}}$  instead of  $\rho(u, u')^{\frac{\alpha}{M}}$  in (2.4.1) (the proof is similar, see Remark 2.3.11).

*Proof of Theorem 2.4.1.* 1<sup>ST</sup> STEP. Let  $w = w_1$  and  $w' = w'_1$  as it was earlier. In the frame  $\{\widehat{X}_i^u\}_{i=1}^N$  we have

$$w' = \exp\left(\sum_{i=1}^N w'_i \widehat{X}_i^u\right)(v).$$

Consider the point

$$\omega_\varepsilon = \exp\left(\sum_{i=1}^N w'_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v).$$

Note that  $\omega_1 = w'$ . In view of the generalized triangle inequality,  $d_\infty^u(w_\varepsilon, w'_\varepsilon) \leq c(d_\infty^u(w_\varepsilon, \omega_\varepsilon) + d_\infty^u(\omega_\varepsilon, w'_\varepsilon))$ . By the above estimate

$$d_\infty^u(\omega_\varepsilon, w_\varepsilon) = \varepsilon d_\infty^u(w, w') = \varepsilon \Theta(\rho(u, u')^\alpha d_\infty^u(v, w))^{\frac{1}{M}}. \quad (2.4.2)$$

Note that, if  $\alpha = 0$ , then we obtain here  $\omega_\Xi(\rho(u, u'))$ .

Now we estimate the distance  $d_\infty^u(\omega_\varepsilon, w'_\varepsilon)$ . Represent  $w'_\varepsilon$  in the frame  $\{\widehat{X}_i^u\}_{i=1}^N$ :

$$w'_\varepsilon = \exp\left(\sum_{i=1}^N \alpha_i(\varepsilon) \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v), \quad (2.4.3)$$

and consider the point

$$\omega' = \exp\left(\sum_{i=1}^N \alpha_i(\varepsilon) \widehat{X}_i^u\right)(v).$$

Here the coefficients  $\alpha_i(\varepsilon)$ ,  $i = 1, \dots, N$ , depend on  $u$  and  $\{w_i\}_{i=1}^N$ .

2<sup>ND</sup> STEP. Next, we show that the coefficients  $\alpha_i(\varepsilon)$ ,  $i = 1, \dots, N$ , are uniformly bounded for all  $\varepsilon > 0$  uniformly on  $u$  and  $\{w_i\}_{i=1}^N$ . By another words, there exists  $S < \infty$  such that  $d_\infty^u(v, w'_\varepsilon) \leq S\varepsilon$  for all  $\varepsilon > 0$  small enough and all  $u$  and  $\{w_i\}_{i=1}^N$ . Indeed, by the generalized triangle inequality for Carnot groups, we have

$$d_\infty^u(v, w'_\varepsilon) \leq c(d_\infty^u(u, v) + d_\infty^u(u, w'_\varepsilon)).$$

Next,  $d_\infty^u(u, w'_\varepsilon) = d_\infty(u, w'_\varepsilon)$ . Since  $d_\infty(u, v) = \mathcal{C}\varepsilon$ , it is enough to show that  $d_\infty(u, w'_\varepsilon) \leq K\varepsilon$ . To do this, we estimate the value  $d_\infty(u', w'_\varepsilon)$ . Since  $d_\infty(u', w'_\varepsilon) = d_\infty^{u'}(u', w'_\varepsilon)$ , then in view of the generalized triangle inequality for Carnot groups, we have

$$d_\infty^{u'}(u', w'_\varepsilon) \leq c(d_\infty^{u'}(u', v) + d_\infty^{u'}(v, w'_\varepsilon)).$$

The conditions  $d_\infty(u, u') = C\varepsilon$ ,  $d_\infty(u, v) = \mathcal{C}\varepsilon$  and Theorem 2.2.13 imply

$$d_\infty^{u'}(u', v) = d_\infty(u', v) \leq L \max\{C, \mathcal{C}\}\varepsilon.$$

Applying Theorem 2.2.13 again, we infer

$$d_\infty(u, w'_\varepsilon) \leq K\varepsilon.$$

From here and from the fact that  $d_\infty(u, v) = \mathcal{C}\varepsilon$ , we have

$$d_\infty^u(v, w'_\varepsilon) \leq S\varepsilon$$

for all  $\varepsilon > 0$  small enough and all  $u$  and  $\{w_i\}_{i=1}^N$  belonging to some compact neighborhoods.

From here, we have that all  $\alpha_i(\varepsilon)$ ,  $i = 1, \dots, N$ , are bounded uniformly in  $\varepsilon > 0$ .



3<sup>RD</sup> STEP. Note that  $d_\infty^u(\omega_\varepsilon, w'_\varepsilon) = \varepsilon d_\infty^u(\omega', w')$ . Consider the mapping

$$\Delta_{\varepsilon, v}^u(x) = \exp\left(\sum_{i=1}^N x_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v). \quad (2.4.4)$$

More exactly,

$$\begin{aligned} \mathbb{M} \ni x \mapsto \{x_1, \dots, x_N\} \text{ by such a way that } x &= \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(v) \\ &\xrightarrow{\Delta_{\varepsilon, v}^u} \exp\left(\sum_{i=1}^N x_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v). \end{aligned} \quad (2.4.5)$$

Show that the coordinate functions are  $H^\alpha$ -continuous in  $u \in \mathbb{M}$  uniformly on  $\varepsilon > 0$ .

**1. The case of  $\alpha > 0$ .** Indeed, the mapping

$$\theta_{v, u}(x_1, \dots, x_N) = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(v),$$

where  $(x_1, \dots, x_N) \in \text{Box}(0, T\varepsilon)$ , is  $H^\alpha$ -continuous in  $u \in \mathbb{M}$  as a solution to an equation with  $H^\alpha$ -continuous right-hand part (see Section 5), and its Hölder constant does not depend on  $v$  belonging to some compact set. This mapping is also quasi-isometric on  $(x_1 \varepsilon^{-\deg X_1}, \dots, x_N \varepsilon^{-\deg X_N}) \in \mathbb{R}^N$  with respect to the Riemannian metric. Consider now the inverse mapping, which assigns to a given point  $x \in \mathbb{M}$ ,  $d_\infty^u(v, x) \leq T\varepsilon$ , the “coordinates”  $x_1(u, x) \varepsilon^{-\deg X_1}, \dots, x_N(u, x) \varepsilon^{-\deg X_N}$  such that

$$x = \exp\left(\sum_{i=1}^N x_i(u, x) \widehat{X}_i^u\right)(v).$$

Note that the quasi-isometric coefficients of the mapping  $\theta_{v, u}$  are independent from  $(x_1, \dots, x_N)$ ,  $u$  and  $v$  belonging to some compact set (here we suppose that  $d_\infty^u(v, x) \leq T\varepsilon$ ). Show that the functions  $x_1(u, x) \varepsilon^{-\deg X_1}, \dots, x_N(u, x) \varepsilon^{-\deg X_N}$  are  $H^\alpha$ -continuous in  $u \in U$  for a fixed  $x \in \mathbb{M}$ , and their Hölder constants are bounded locally uniformly in  $x$ ,  $v$  and in  $\varepsilon > 0$ . (Here, to guarantee the uniform boundedness of  $x_1(u, x) \varepsilon^{-\deg X_1}, \dots, x_N(u, x) \varepsilon^{-\deg X_N}$ , we assume that

- both values  $d_\infty(u, v)$  and  $d_\infty^u(v, x)$  are comparable to  $\varepsilon$
- the point  $u$  can be changed only by a point  $u'$ , such that the distance  $d_\infty(u, u')$  is also comparable to  $\varepsilon$  (see 2<sup>nd</sup> step.)

The latter statement follows from the fact, that  $\theta_{u,v}(x_1, \dots, x_N)$  is locally Hölder in  $u$ , and its Hölder constant is independent of  $v$  belonging to some compact set, and of  $(x_1, \dots, x_N)$  belonging to some compact neighborhood  $U(0)$  of zero. Since we prove a local property of a mapping then we may assume that  $u, u', x$  and  $v$  meet our above condition on  $d_\infty$ -distances and they belong to some compact neighborhood  $U$  such that the mapping  $\theta_{u,v}$  is bi-Lipschitz on  $(x_1 \varepsilon^{-\deg X_1}, \dots, x_N \varepsilon^{-\deg X_N})$  if  $u \in U$ ; moreover, its bi-Lipschitz coefficients are independent of  $u, (x_1 \varepsilon^{-\deg X_1}, \dots, x_N \varepsilon^{-\deg X_N})$  and  $v$  belonging to some compact set. Indeed, consider the mapping  $\theta_v(u, x_1, \dots, x_N) = \theta_{u,v}(x_1, \dots, x_N)$  and suppose that for any  $L > 0$  there exist  $\varepsilon > 0$ , points  $v, x \in U$ , a level set  $\theta_v^{-1}(x)$ , and points  $(u, x_1(u), \dots, x_N(u))$  and  $(u', x_1(u'), \dots, x_N(u'))$  on it such that

$$\begin{aligned} & \left| (x_1(u) \varepsilon^{-\deg X_1}, \dots, x_N(u) \varepsilon^{-\deg X_N}) \right. \\ & \quad \left. - (x_1(u') \varepsilon^{-\deg X_1}, \dots, x_N(u') \varepsilon^{-\deg X_N}) \right| \geq L |u - u'|^\alpha \quad (2.4.6) \end{aligned}$$

for some  $u$  and  $u'$ . The assumption (2.4.6) leads to the following contradiction:

$$\begin{aligned} 0 &= \left| \theta_v(u, x_1(u) \varepsilon^{-\deg X_1}, \dots, x_N(u) \varepsilon^{-\deg X_N}) \right. \\ & \quad \left. - \theta_v(u', x_1(u') \varepsilon^{-\deg X_1}, \dots, x_N(u') \varepsilon^{-\deg X_N}) \right| \\ &\geq \left| \theta_v(u, x_1(u) \varepsilon^{-\deg X_1}, \dots, x_N(u) \varepsilon^{-\deg X_N}) \right. \\ & \quad \left. - \theta_v(u, x_1(u') \varepsilon^{-\deg X_1}, \dots, x_N(u') \varepsilon^{-\deg X_N}) \right| \\ &\quad - \left| \theta_v(u, x_1(u') \varepsilon^{-\deg X_1}, \dots, x_N(u') \varepsilon^{-\deg X_N}) \right. \\ & \quad \left. - \theta_v(u', x_1(u') \varepsilon^{-\deg X_1}, \dots, x_N(u') \varepsilon^{-\deg X_N}) \right| \\ &\geq C_x \left| (x_1(u) \varepsilon^{-\deg X_1}, \dots, x_N(u) \varepsilon^{-\deg X_N}) \right. \\ & \quad \left. - (x_1(u') \varepsilon^{-\deg X_1}, \dots, x_N(u') \varepsilon^{-\deg X_N}) \right| \\ &\quad - C_u |u - u'|^\alpha \geq (LC_x - C_u) |u - u'|^\alpha > 0 \quad (2.4.7) \end{aligned}$$

if  $L > \frac{C_u}{C_x}$ .

Note that  $\omega' = \Delta_{\varepsilon^{-1},v}^u(\Delta_{\varepsilon,v}^{u'}(w'))$ , and  $w' = \Delta_{\varepsilon^{-1},v}^{u'}(\Delta_{\varepsilon,v}^{u'}(w'))$ . Here, for the point  $w'_\varepsilon = \Delta_{\varepsilon,v}^{u'}(w')$ , we have  $x_i(u, w'_\varepsilon) = \alpha_i(\varepsilon) \cdot \varepsilon^{\deg X_i}$  on the one hand, and we have  $x_i(u', w'_\varepsilon) = w_i \cdot \varepsilon^{\deg X_i}$  on the other hand,  $i = 1, \dots, N$ . Since the points  $u, u', v$  and  $w'_\varepsilon$  meet our assumption on points, we have that the Hölder

constants of  $x_i(u, x)\varepsilon^{-\deg X_i}$  are bounded uniformly in  $\{w_j\}_{j=1}^N$  belonging to some neighborhood of zero. Hence,  $\rho(\omega', w') = \Theta\rho(u, u')^\alpha$ , and

$$d_\infty^u(\omega', w') = \Theta\rho(u, u')^{\frac{\alpha}{M}}. \quad (2.4.8)$$

**2. The case of  $\alpha = 0$**  is proved similarly to the previous case. We prove that the functions  $x_1(u, x)\varepsilon^{-\deg X_1}, \dots, x_N(u, x)\varepsilon^{-\deg X_N}$  are uniformly continuous in  $u \in U$  for a fixed  $x \in \mathbb{M}$ , and this continuity is uniform in  $x, v$  and  $\varepsilon > 0$ . The points under consideration meet the above condition.

To prove our result, we assume the contrary that there exists  $\sigma > 0$  such that for any  $\delta > 0$  there exist  $\varepsilon > 0$ , points  $v, x \in U$ , a level set  $\theta_v^{-1}(x)$ , and points  $(u, x_1(u), \dots, x_N(u))$  and  $(u', x_1(u'), \dots, x_N(u'))$  on it such that  $|u - u'| < \delta$ , and in the right-hand part of (2.4.6) instead of  $L|u - u'|^\alpha$ , we obtain  $\sigma$ .

Repeating further the scheme of the proof almost verbatim and replacing  $(LC_x - C_u)|u - u'|^\alpha$  by  $\sigma C_x - \omega_{\theta_v}(u)$  in the right-hand part of (2.4.7), we deduce

$$\rho(\omega', w') = \omega_{\Delta_{\varepsilon^{-1}, v}^u \circ \Delta_{\varepsilon, v}^{u'}}(\rho(u, u')). \quad (2.4.9)$$

We may assume without loss of generality, that  $\omega_{\Delta_{\varepsilon^{-1}, v}^u \circ \Delta_{\varepsilon, v}^{u'}}$  does not depend on  $x$  and  $v$  (see (2.4.6) and (2.4.7)).

4<sup>TH</sup> STEP. Taking (2.4.2), (2.4.8) and (2.4.9) into account we obtain

$$d_\infty^u(w_\varepsilon, w'_\varepsilon) = \varepsilon[\Theta(C, \mathcal{C})]\rho(u, u')^{\frac{\alpha}{M}}$$

for  $\alpha > 0$ . Similarly, we obtain the theorem for  $\alpha = 0$ . The theorem follows.  $\square$

**Corollary 2.4.3. 1.** *Note that  $d_\infty(u, u') = C\varepsilon$  implies  $\rho(u, u') < C\varepsilon$ . Then, for  $\alpha > 0$ , we have*

$$d_\infty^u(w_\varepsilon, w'_\varepsilon) = O(\varepsilon^{1+\frac{\alpha}{M}}) \text{ as } \varepsilon \rightarrow 0$$

where  $O$  is uniform in  $u, u', v \in U \subset \mathbb{M}$ , and in  $\{w_i\}_{i=1}^N$  belonging to some compact neighborhood of 0, and depends on  $C$  and  $\mathcal{C}$ .

**2.** *If  $\alpha = 0$  then*

$$d_\infty^u(w_\varepsilon, w'_\varepsilon) = o(\varepsilon) \text{ as } \varepsilon \rightarrow 0$$

where  $o$  is uniform in  $u, u', v \in U \subset \mathbb{M}$ , and in  $\{w_i\}_{i=1}^N$  belonging to some compact neighborhood of 0, and depends on  $C$  and  $\mathcal{C}$ .

**Remark 2.4.4.** The estimate  $O(\varepsilon^{1+\frac{\alpha}{M}})$  is also true for the case of vector fields  $X_i, i = 1, \dots, N$ , which are Hölder with respect to such  $\mathfrak{d}$  that  $d_\infty(u, u') = C\varepsilon$  implies  $\mathfrak{d}(u, u') = K\varepsilon$ , where  $K$  is bounded for  $u, u' \in U$ .

A particular case is  $\mathfrak{d} = d_\infty^z$ , where  $d_\infty(z, u) \leq Q\varepsilon$  (see Local Approximation Theorem 2.5.4, case  $\alpha = 0$ , below).

**Remark 2.4.5.** The estimate  $O(\varepsilon^{1+\frac{\alpha}{M}})$  is also true for the case of vector fields  $X_i, i = 1, \dots, N$ , which are Hölder with respect to such  $\mathfrak{d}$  that  $d_\infty(u, u') = C\varepsilon$  implies  $\mathfrak{d}(u, u') = K\varepsilon$ , where  $K$  is bounded for  $u, u' \in U$ .

A particular case is  $\mathfrak{d} = d_\infty^z$ , where  $d_\infty(z, u) \leq Q\varepsilon$  (see Local Approximation Theorem 2.5.4, case  $\alpha = 0$ , below).

## 2.5 The Approximation Theorems

In this subsection, we prove two Approximation Theorems. Their proofs use the following geometric property.

**Proposition 2.5.1.** *For a neighborhood  $U$ , there exist positive constants  $C > 0$  and  $r_0 > 0$  depending on  $U$ ,  $M$ , and  $N$ , such that for any points  $u$  and  $v$  from a neighborhood  $U$  the following inclusion is valid:*

$$\bigcup_{x \in \text{Box}^u(v, r)} \text{Box}^u(x, \xi) \subseteq \text{Box}^u(v, r + C\xi), \quad 0 < \xi, r \leq r_0.$$

*Proof.* Let  $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(v)$ ,  $d_\infty^u(v, x) \leq r$ , and  $z = \exp\left(\sum_{i=1}^N z_i \widehat{X}_i^u\right)(x)$ ,  $d_\infty^u(x, z) \leq \xi$ . We estimate the distance  $d_\infty^u(v, z)$  applying (2.1.4) to points  $x$  and  $z$ . Let  $z = \exp\left(\sum_{i=1}^N \zeta_i \widehat{X}_i^u\right)(v)$ .

CASE of  $\deg X_i = 1$ . Then  $|\zeta_i| \leq |x_i| + |z_i| \leq (r + \xi)^{\deg X_i}$ .

CASE of  $\deg X_i = 2$ . Then

$$\begin{aligned} |\zeta_i| &\leq |x_i| + |z_i| + \sum_{\substack{|e_l + e_j|_h = 2, \\ l < j}} |F_{e_l, e_j}^i(u)| |x_l z_j - z_l x_j| \\ &\leq r^2 + \xi^2 + c_i(u) r \xi \leq r^2 + 2r \frac{c_i(u)}{2} \xi + \left(\frac{c_i(u)}{2} \xi\right)^2 \\ &= \left(r + \frac{c_i(u)}{2} \xi\right)^{\deg X_i} = (r + C_i(u) \xi)^{\deg X_i}. \end{aligned}$$

Here we assume that  $C_i(u) \geq 1$ .

CASE of  $\deg X_i = k > 2$ . Then we obtain analogously to the previous case

$$\begin{aligned} |\zeta_i| &\leq |x_i| + |z_i| + \sum_{|\mu+\beta|_h=k, \mu>0, \beta>0} |F_{\mu,\beta}^i(u)| x^\mu \cdot z^\beta \\ &\leq r^k + \xi^k + \sum_{|\mu+\beta|_h=k} c_i^{\mu\beta}(u) r^{|\mu|_h} \xi^{|\beta|_h} \leq (r + C_i(u)\xi)^{\deg X_i}. \end{aligned}$$

Here we assume that  $C_i(u), c_i(u) \geq 1$ . Denote by  $C(u) = \max_i C_i(u)$ . From above estimates we obtain

$$d_\infty^u(v, x) = \max_i \{|\zeta_i|^{\deg X_i}\} \leq \max_i \{(r + C_i(u)\xi)^{\frac{\deg X_i}{\deg X_i}}\} \leq r + C(u)\xi.$$

Since all the  $C_i(u)$ 's are continuous on  $u$  then we may choose  $C < \infty$  such that  $C(u) \leq C$  for all  $u$  belonging to a compact neighborhood. The lemma follows.  $\square$

**Theorem 2.5.2** (Approximation Theorem). *Let  $u, u', v, w \in U$ . Then the following estimate is valid:*

$$|d_\infty^u(v, w) - d_\infty^{u'}(v, w)| = \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}. \quad (2.5.1)$$

*Proof.* Let  $p = \exp\left(\sum_{i=1}^N p_i \widehat{X}_i^u\right)(v)$  and  $p' = \exp\left(\sum_{i=1}^N p_i \widehat{X}_i^{u'}\right)(v)$ . Notice that if  $z \in \text{Box}^u(v, d_\infty^u(v, w))$  then  $z' \in \text{Box}^{u'}(v, d_\infty^u(v, w))$  and  $z \in \text{Box}^{u'}(z', R(u, u'))$ , where

$$R(u, u') = \sup_{p' \in \text{Box}^{u'}(v, d_\infty^u(v, w))} d_\infty^{u'}(p, p').$$

Using Proposition 2.5.1 we have that

$$\begin{aligned} \text{Box}^u(v, d_\infty^u(v, w)) &\subset \bigcup_{x \in \text{Box}^{u'}(v, d_\infty^u(v, w))} \text{Box}^{u'}(x, R(u, u')) \\ &\subset \text{Box}^{u'}(v, d_\infty^u(v, w) + CR(u, u')) \end{aligned}$$

for some  $C > 0$ . Consequently, in view of Theorem 2.3.9 we can write

$$\begin{aligned} \text{Box}^u(v, d_\infty^u(v, w)) &\subset \text{Box}^{u'}(v, d_\infty^u(v, w) + CR(u, u')) \subset \\ &\text{Box}^{u'}(v, d_\infty^u(v, w) + \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}). \end{aligned}$$

If  $d_\infty^u(v, w) \leq \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}$  then the theorem follows:

$$|d_\infty^u(v, w) - d_\infty^{u'}(v, w)| \leq d_\infty^u(v, w) + d_\infty^{u'}(v, w) = \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}.$$

If  $d_\infty^u(v, w) > \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}$  then applying again Proposition 2.5.1 we obtain

$$\text{Box}^{u'}(v, d_\infty^u(v, w) - \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}) \subset \text{Box}^u(v, d_\infty^u(v, w)).$$

From the latter relation it follows that

$$\begin{aligned} d_\infty^u(v, w) - \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}} &\leq d_\infty^{u'}(v, w) \\ &\leq d_\infty^u(v, w) + \Theta[\rho(u, u')^\alpha \rho(v, w)]^{\frac{1}{M}}, \end{aligned}$$

and the theorem follows.  $\square$

**Remark 2.5.3.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , then we have  $\mathfrak{d}(u, u')^\alpha$  instead of  $\rho(u, u')^\alpha$  in (2.5.1) (the proof is similar).

Approximation Theorem and local estimates (see Theorem 2.4.1) imply Local Approximation Theorem.

**Theorem 2.5.4** (Local Approximation Theorem). *Assume that  $d_\infty(u, u') = C\varepsilon$ ,  $d_\infty(u, v) = \mathcal{C}\varepsilon$  and  $d_\infty(u, w) = \mathbb{C}\varepsilon$  for some  $C, \mathcal{C}, \mathbb{C} < \infty$ .*

1. *If  $\alpha > 0$ , then*

$$|d_\infty^u(v, w) - d_\infty^{u'}(v, w)| = \varepsilon \Theta[\rho(u, u')]^{\frac{\alpha}{M}} \Theta(d_\infty^u(v, w) + o(1)). \quad (2.5.2)$$

*Moreover, if  $u' = v$  and  $\alpha > 0$ , then*

$$|d_\infty^u(v, w) - d_\infty(v, w)| = \varepsilon \Theta[\rho(u, v)]^{\frac{\alpha}{M}} \Theta(d_\infty^u(v, w) + o(1)).$$

2. *If  $\alpha = 0$ , then*

$$|d_\infty^u(v, w) - d_\infty^{u'}(v, w)| = \varepsilon o(1) = o(\varepsilon)$$

*where  $o$  is uniform in  $u, u', v, w \in U \subset \mathbb{M}$ . Moreover, if  $u' = v$  and  $\alpha = 0$ , then*

$$|d_\infty^u(v, w) - d_\infty(v, w)| = o(\varepsilon)$$

*where  $o$  is uniform in  $u, v, w \in U \subset \mathbb{M}$ .*

*Proof* follows the scheme as the proof of Approximation Theorem 2.5.2 with  $R(u, u') = \varepsilon[\Theta(C, \mathcal{C}, \mathbb{C})]\rho(u, u')^{\frac{\alpha}{M}}$ . The latter equality is valid by the uniformity assertion of Theorem 2.4.1.

**Remark 2.5.5.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , then we have  $\mathfrak{d}(u, u')^{\frac{\alpha}{M}}$  instead of  $\rho(u, u')^{\frac{\alpha}{M}}$  in (2.5.2) (the proof is similar).

## 2.6 Comparison of Local Geometries of Two Local Carnot Groups

*Proof of Theorem 2.3.1.* 1<sup>ST</sup> STEP. Consider the case of  $\alpha > 0$ . The case of  $Q = 1$  is proved in Theorem 2.4.1.

2<sup>ND</sup> STEP. Consider the case of  $Q = 2$ . First, for the points  $w_2 = w_2^1$  and  $w_2' = w_2^{1'}$ , we have

$$w_2 = \exp\left(\sum_{i=1}^N \omega_{i,2} \widehat{X}_i^u\right)(w_0) \quad (2.6.1)$$

and

$$w_2' = \exp\left(\sum_{i=1}^N \omega_{i,2}' \widehat{X}_i^{u'}\right)(w_0). \quad (2.6.2)$$

By the formulas of group operation,  $\omega_{i,2}$  differs from  $\omega_{i,2}'$  in the values of  $\{F_{\mu,\beta}^j(u)\}_{j,\mu,\beta}$ . By Assumption 2.1.4,  $F_{\mu,\beta}^j(u') = F_{\mu,\beta}^j(u) + \Theta\rho(u, u')^\alpha$ .

Consider the auxiliary points

$$w_2'' = \exp\left(\sum_{i=1}^N \omega_{i,2} \widehat{X}_i^{u'}\right)(w_0) \text{ and } w_2''^\varepsilon = \exp\left(\sum_{i=1}^N \omega_{i,2} \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\right)(w_0)$$

and estimate the value  $d_\infty^{u'}(w_2'', w_2')$ . For doing this, we use the group operation in the local Carnot group  $\mathcal{G}^{u'}\mathbb{M}$  and Approximation Theorem 2.5.2. Note that,  $|\omega_{i,2} - \omega_{i,2}'| = \Theta\rho(u, u')^\alpha$ . Next, note that while applying the group operation, all summands look like  $\omega_{i,2} - \omega_{i,2}'$  or  $\omega_{i,2} - \omega_{i,2}' + \sum \Theta(\omega_{k,2}\omega_{j,2}' - \omega_{j,2}\omega_{k,2}')$ . By (2.1.4), we deduce

$$\begin{aligned} \omega_{k,2}\omega_{j,2}' - \omega_{j,2}\omega_{k,2}' \\ = \omega_{k,2}(\omega_{j,2} + \Theta\rho(u, u')^\alpha) - \omega_{j,2}(\omega_{k,2} + \Theta\rho(u, u')^\alpha) = \Theta\rho(u, u')^\alpha, \end{aligned}$$

$d_\infty^{u'}(w_2'', w_2') = \Theta(\rho(u, u')^{\frac{\alpha}{M}})$ . Here  $\Theta$  depends on  $C, \mathcal{C}, Q = 2$  and  $\{F_{\mu,\beta}^j(u')\}_{j,\mu,\beta}$ .

It follows from the formulas of group operation in  $\mathcal{G}^u\mathbb{M}$  and  $\mathcal{G}^{u'}\mathbb{M}$ , that

$$w_2^\varepsilon = \exp\left(\sum_{i=1}^N \omega_{i,2} \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(w_0)$$

and

$$w_2^{\varepsilon'} = \exp\left(\sum_{i=1}^N \omega_{i,2}' \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\right)(w_0).$$

By Theorem 2.4.1, we have  $d_\infty^{u'}(w_2^{\varepsilon'}, w_2^\varepsilon) = \varepsilon \Theta \rho(u, u')^{\frac{\alpha}{M}}$ . By the homogeneity of the distance  $d_\infty^{u'}$  we have

$$d_\infty^{u'}(w_2^{\varepsilon'}, w_2^{\varepsilon'}) = \varepsilon d_\infty^{u'}(w_2'', w_2') = \varepsilon \Theta \rho(u, u')^{\frac{\alpha}{M}},$$

and from the generalized triangle inequality we deduce

$$d_\infty^{u'}(w_2^\varepsilon, w_2^{\varepsilon'}) = \varepsilon \Theta \rho(u, u')^{\frac{\alpha}{M}}.$$

In view of Local Approximation Theorem 2.5.4, we derive

$$d_\infty^u(w_2^\varepsilon, w_2^{\varepsilon'}) = \varepsilon \Theta \rho(u, u')^{\frac{\alpha}{M}}.$$

3<sup>RD</sup> STEP. In the case of  $Q = 3$ , it is easy to see from the previous case and the group operation, that if

$$w_3 = \exp\left(\sum_{i=1}^N \omega_{i,3} \widehat{X}_i^u\right)(w_0)$$

and

$$w'_3 = \exp\left(\sum_{i=1}^N \omega'_{i,3} \widehat{X}_i^{u'}\right)(w_0),$$

then again  $|\omega_{i,3} - \omega'_{i,3}| = \Theta \rho(u, u')^\alpha$ . Here  $\Theta$  depends on  $C, \mathcal{C}, Q = 3$  and  $\{F_{\mu,\beta}^j\}_{j,\mu,\beta}$ . (It suffices to apply the group operation in local Carnot groups  $\mathcal{G}^u \mathbb{M}$  and  $\mathcal{G}^{u'} \mathbb{M}$  to expressions (2.6.1) and (2.6.2) and to points  $w_3$  and  $w'_3$ , respectively.) From now on, for obtaining estimate (2.3.1) at  $Q = 3$ , we repeat the arguments of the 2<sup>nd</sup> Step.

4<sup>TH</sup> STEP. It is easy to see analogously to the 3<sup>rd</sup> Step, that the group operation and the induction hypothesis  $|\omega_{i,l-1} - \omega'_{i,l-1}| = \Theta \rho(u, u')^\alpha$ ,  $3 < l < Q$ , imply  $|\omega_{i,l} - \omega'_{i,l}| = \Theta \rho(u, u')^\alpha$ . Indeed, it suffices to put  $\omega_{i,l}$  and  $\omega'_{i,l}$  instead of  $\omega_{i,3}$  and  $\omega'_{i,3}$ , and  $\omega_{i,l-1}$  and  $\omega'_{i,l-1}$  instead of  $\omega_{i,2}$  and  $\omega'_{i,2}$  in the 3<sup>rd</sup> Step, and apply arguments from the 2<sup>nd</sup> Step.

The case of  $\alpha = 0$  can be proved by applying the similar arguments.

The theorem follows.  $\square$

## 2.7 Comparison of Local Geometries of a Carnot Manifold and a Local Carnot Group

In this subsection, we compare the local geometry of a Carnot manifold with the one of a local Carnot group.



**Theorem 2.7.1.** Fix  $Q \in \mathbb{N}$ . Consider points  $w_0, u$  such that  $d_\infty(u, w_0) = \mathcal{C}\varepsilon$  for some  $\mathcal{C} < \infty$ , and

$$\widehat{w}_j^\varepsilon = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(\widehat{w}_{j-1}^\varepsilon), \quad w_j^\varepsilon = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} X_i\right)(w_{j-1}^\varepsilon),$$

$w_0^\varepsilon = \widehat{w}_0^\varepsilon = \widehat{w}_0 = w_0$ ,  $j = 1, \dots, Q$ . (Here  $Q \in \mathbb{N}$  is such that all these points belong to a neighborhood  $U \subset \mathbb{M}$  small enough for all  $\varepsilon > 0$ .) Then for  $\alpha > 0$

$$\max\{d_\infty^u(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon), d_\infty(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon)\} = \sum_{k=1}^Q \Theta(\mathcal{C}, k, \{F_{\mu,\beta}^j\}_{j,\mu,\beta}) \cdot \varepsilon^{1+\frac{\alpha}{M}}. \quad (2.7.1)$$

In the case of  $\alpha = 0$  we have

$$\{d_\infty^u(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon), d_\infty(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon)\} = \varepsilon \cdot \Theta(\mathcal{C}, Q, \{F_{\mu,\beta}^j\}_{j,\mu,\beta})[\omega(\varepsilon)]^{\frac{1}{M}}$$

where  $\omega(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Here  $|w_{i,j}|$  are bounded, and  $\Theta$  is uniformly bounded for  $u, w_0 \in U$  and  $\{w_{i,j}\}$ ,  $i = 1, \dots, N$ ,  $j = 1, \dots, Q$ , belonging to some compact neighborhood of 0, and it depends on  $Q$  and  $\{F_{\mu,\beta}^j\}_{j,\mu,\beta}$ .

*Proof.* For simplifying the notation we denote the points  $\widehat{w}_i^1$  by  $\widehat{w}_i$ , and we denote  $w_i^1$  by  $w_i$  for  $\varepsilon = 1$ . First, consider the points  $\widehat{w}_Q$  and  $w_Q$ . Now we construct a following sequence of points.

Let

$$\omega_{k,j} = \exp\left(\sum_{i=1}^N w_{i,j} \widehat{X}_i^{w_k}\right)(\omega_{k,j-1}),$$

$$k = 0, \dots, Q-1, \quad j = 1, \dots, Q-k, \quad \omega_{k,0} = w_k.$$

Hence,  $\omega_{Q-1,1} = w_Q$  and

$$d_\infty^u(w_Q, w'_Q) = O\left(d_\infty^u(w_Q, \omega_{0,Q}) + \sum_{k=1}^{Q-1} d_\infty^u(\omega_{k,Q-k}, \omega_{k-1,Q-k+1})\right).$$

If  $\alpha > 0$  then, by Theorem 2.3.1,

$$d_\infty^u(\widehat{w}_Q, \omega_{0,Q}) = \Theta(\mathcal{C}, Q, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta}) \rho(u, w_0)^{\frac{\alpha}{M}},$$

and each of the summands

$$d_\infty^u(\omega_{k,Q-k}, \omega_{k-1,Q-k+1}) = \Theta(\mathcal{C}, Q-k, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta}) \rho(w_k, w_{k-1})^{\frac{\alpha}{M}}.$$

By the same theorem, if we replace  $w_{i,j}$  by  $w_{i,j}\varepsilon^{\deg X_i}$  then it is easy to see using induction by  $k$  that firstly  $d_\infty^u(w_k^\varepsilon, w_{k-1}^\varepsilon) = O(\varepsilon)$ , secondly  $d_\infty^u(u, w_k^\varepsilon) \sim \varepsilon$  and  $d_\infty^u(u, \omega_{k,Q-k}^\varepsilon) \sim \varepsilon$  for all  $k$ , and thirdly

$$d_\infty^u(\widehat{w}_Q^\varepsilon, \omega_{0,Q}^\varepsilon) = \varepsilon \Theta(\mathcal{C}, Q, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta}) \rho(u, w_0)^{\frac{\alpha}{M}}$$

and

$$d_\infty^u(\omega_{k,Q-k}^\varepsilon, \omega_{k-1,Q-k+1}^\varepsilon) = \varepsilon \Theta(\mathcal{C}, Q - k, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta}) \rho(w_k^\varepsilon, w_{k-1}^\varepsilon)^{\frac{\alpha}{M}}.$$

Thus we obtain  $d_\infty^u(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon) = \sum_{k=1}^Q \Theta(\mathcal{C}, k, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta}) \cdot \varepsilon^{1+\frac{\alpha}{M}}$ .

Since  $d_\infty^u(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon) = O(\varepsilon)$  and  $d_\infty^u(u, \widehat{w}_Q^\varepsilon) = O(\varepsilon)$  then, by Local Approximation Theorem 2.5.4, we have

$$d_\infty(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon) = \sum_{k=1}^Q \Theta(\mathcal{C}, k, \{F_{\alpha,\beta}^j\}_{j,\alpha,\beta}) \cdot \varepsilon^{1+\frac{\alpha}{M}}.$$

If  $\alpha = 0$ , then we repeat the above arguments replacing  $\rho(\cdot, \cdot)^{\frac{\alpha}{M}}$  by  $o(1)$ . The theorem follows.  $\square$

**Remark 2.7.2.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , such that  $d_\infty(x, y) \leq \varepsilon$  implies  $\mathfrak{d}(x, y) \leq K\varepsilon$ , where  $K$  is bounded on  $U$ , then the same estimate as in (2.7.1) is true (the proof is similar).

A particular case of such  $\mathfrak{d}$  is  $d_\infty^z$ ,  $d_\infty(u, z) \leq Q\varepsilon$  (see Local Approximation Theorem 2.5.4).

## 2.8 Applications

### 2.8.1 Rashevskii–Chow Theorem

**Definition 2.8.1.** An absolutely continuous curve  $\gamma : [0, a] \rightarrow \mathbb{M}$  is said to be *horizontal* if  $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$  for almost all  $t \in [0, a]$ . Its length  $l(\gamma)$  equals  $\int_0^a |\dot{\gamma}(t)| dt$ , where the value  $|\dot{\gamma}(t)|$  is calculated using the Riemann tensor on  $\mathbb{M}$ . Analogously, the canonical Riemann tensor on  $\mathcal{G}^u\mathbb{M}$  defines a length  $\widehat{l}$  of an absolutely continuous curve  $\widehat{\gamma} : [0, a] \rightarrow \mathcal{G}^u\mathbb{M}$ .

**Definition 2.8.2.** The *Carnot–Carathéodory distance* between points  $x, y \in \mathbb{M}$  is defined as  $d_c(x, y) = \inf_{\gamma} l(\gamma)$  where the infimum is taken over all horizontal curves with endpoints  $x$  and  $y$ .

**Corollary 2.8.3** (of Theorem 2.7.1). *Suppose that Assumption 2.1.4 holds for  $\alpha \in (0, 1]$ . Let  $g \in \mathbb{M}$ . Let also  $\varepsilon$  be small enough, and  $u, v, \widehat{w} \in \text{Box}(g, \varepsilon)$ . The points  $v, \widehat{w} \in \text{Box}(g, \varepsilon)$  can be joined in the local Carnot group  $(\mathcal{G}^u \mathbb{M}, d_1^u) \supset \text{Box}(g, \varepsilon)$  by a horizontal curve  $\widehat{\gamma}$  composed by at most  $L$  segments of integral curves of horizontal fields  $\widehat{X}_i^u$ ,  $i = 1, \dots, \dim H_1$ . To the curve  $\widehat{\gamma}$  it corresponds a curve  $\gamma$ , horizontal with respect to the initial horizontal distribution  $H\mathbb{M}$ , constituted by at most  $L$  segments of integral curves of the given horizontal fields  $X_i$ ,  $i = 1, \dots, n$ . Moreover,*

1. *the curve  $\gamma$  has endpoints  $v, w \in \text{Box}(g, O(\varepsilon))$ ;*
2.  *$|l(\gamma) - \widehat{l}(\widehat{\gamma})| = o(\widehat{l}(\widehat{\gamma}))$ ;*
3.  *$\max\{d_\infty^u(\widehat{w}, w), d_\infty(\widehat{w}, w)\} \leq C\varepsilon^{1+\frac{\alpha}{M}}$  where  $C$  is independent of  $g, u, v, \widehat{w}$  in some compact set.*

*Proof.* The desired curve comes from those on any Carnot group [46]: given a Carnot group  $\mathbb{G}$  with the vector fields  $\widehat{X}_1, \dots, \widehat{X}_N$ , each point  $x$  can be joint with 0 by a horizontal curve  $\gamma$  constituted by at most  $L$  segments  $\gamma_j$ ,  $j = 1, \dots, L$ , of integral curves of the given basic horizontal vector fields  $\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_{\dim H_1}$ , i. e.,

$$\begin{cases} \dot{\gamma}_1(t) = a_1 \widehat{X}_{i_1}(\gamma_1(t)) \\ \gamma_1(0) = 0, \end{cases}$$

$$\begin{cases} \dot{\gamma}_j(t) = a_j \widehat{X}_{i_j}(\gamma_j(t)) \\ \gamma_j(0) = \gamma_{j-1}(1), \end{cases}$$

$j = 2, \dots, L$ , and from here we have  $x = \gamma_{i_L}(1)$ . By another words,

$$x = \exp(a_L \widehat{X}_{i_L}) \circ \dots \circ \exp(a_1 \widehat{X}_{i_1}), \quad i_j = 1, \dots, \dim H_1,$$

where  $|a_j|$  is controlled by the distance  $d_c(0, x)$ ,  $j = 1, \dots, L$ , and  $L$  is independent of  $x$ .

Now we carry over a construction described above to the local Carnot group  $(\mathcal{G}^u \mathbb{M}, d_1^u) \supset \text{Box}(g, \varepsilon)$ : the given points  $\widehat{w}, v \in \mathcal{G}^u \mathbb{M}$  can be joint by a horizontal curve  $\widehat{\gamma}$ :

$$w = \exp(a_L \widehat{X}_{i_L}) \circ \dots \circ \exp(a_1 \widehat{X}_{i_1})(v), \quad i_j = 1, \dots, \dim H_1, \quad (2.8.1)$$

$j = 1, \dots, L$ . Then the curve  $\gamma$  defined as

$$w = \exp(a_L X_{i_L}) \circ \dots \circ \exp(a_1 X_{i_1})(v), \quad i_j = 1, \dots, \dim H_1, \quad (2.8.2)$$

is horizontal and its length equals  $\widehat{l}(\widehat{\gamma})(1 + o(1))$ . The estimate

$$\max\{d_\infty^u(w, w'), d_\infty(w, w')\} \leq C\varepsilon^{1+\frac{\alpha}{M}}$$

follows immediately from (2.7.1).  $\square$

**Theorem 2.8.4.** *Suppose that Assumption 2.1.4 holds for some  $\alpha \in (0, 1]$ . Let  $g \in \mathbb{M}$ . Given two points  $w, v \in B(g, \varepsilon)$  where  $\varepsilon$  is small enough, there exist a curve  $\gamma$ , horizontal with respect to the initial horizontal distribution  $H\mathbb{M}$ , with endpoints  $w$  and  $v$ , and a horizontal curve  $\widehat{\gamma}$  in the local Carnot group  $(\mathcal{G}^g\mathbb{M}, d_1^g)$  with the same endpoints, such that*

1.  $\widehat{l}(\widehat{\gamma})$  is equivalent to  $d_1^g(w, v)$ ;
2.  $|l(\gamma) - \widehat{l}(\widehat{\gamma})| = o(\widehat{l}(\widehat{\gamma}))$ ;
3. if  $v = g$  then the length  $l(\gamma)$  is equivalent to  $d_\infty(g, w)$ .

All these estimates are uniform in  $w, v$  and  $g$  of some compact neighborhood as  $\varepsilon \rightarrow 0$ .

*Proof.* We can choose  $\varepsilon$  from the condition of the theorem by requests  $C^{2+\frac{\alpha}{M}}\varepsilon^{\frac{\alpha^2}{M^2}} \leq 1$  and  $\varepsilon \leq \frac{1}{2}$ , where  $C$  is the constant from Corollary 2.8.3.

Apply Corollary 2.8.3 to the points  $u = g, v$  and  $w$ . It gives a horizontal curve  $\gamma_1$  ( $\widehat{\gamma}$ ) with respect to the initial horizontal distribution  $H\mathbb{M}$  (in the local Carnot group  $(\mathcal{G}^g\mathbb{M}, d_1^g)$ ) with endpoints  $v$  and  $w_1$  ( $v$  and  $w$ ) constituted by at most  $L$  segments of integral curves of given horizontal fields  $X_i$  ( $\widehat{X}_i^g$ ),  $i = 1, \dots, n$ . The curves  $\widehat{\gamma}$  and  $\gamma_1$  have lengths comparable with  $d_1^g(v, w)$ , and  $\max\{d_\infty^g(w_1, w), d_\infty(w_1, w)\} \leq C\varepsilon^{1+\frac{\alpha}{M}}$ .

Next, we apply again Corollary 2.8.3 to the points  $u = v = w_1$  and  $w$ . It gives a horizontal curve  $\gamma_2$  with respect to  $H\mathbb{M}$  with endpoints  $w_1$  and  $w_2$ . Its length is  $O(\varepsilon^{1+\frac{\alpha}{M}})$  where  $O$  is uniform in  $u, w \in \text{Box}(g, \varepsilon)$ , and  $d_\infty(w_2, w) \leq C(C\varepsilon^{1+\frac{\alpha}{M}})^{1+\frac{\alpha}{M}} \leq \varepsilon^{1+\frac{2\alpha}{M}}$ .

Assume that we have points  $w_1, \dots, w_k$  and horizontal curves  $\gamma_l$ ,  $l = 2, \dots, k$ , with respect to  $H\mathbb{M}$  with endpoints  $w_{l-1}$  and  $w_l$ , such that  $\gamma_l$  has a length  $O(\varepsilon^{1+\frac{l-1}{M}\alpha})$ , and  $d_\infty(w_l, w) \leq \varepsilon^{1+\frac{l\alpha}{M}}$ .

We continue, by the induction, applying Corollary 2.8.3 to the points  $u = v = w_k$  and  $w$ . It results a horizontal curve  $\gamma_{k+1}$  with endpoints  $w_k$  and  $w_{k+1}$ , such that  $\gamma_{k+1}$  has a length  $O(\varepsilon^{1+\frac{k\alpha}{M}})$  and  $d_\infty(w_{k+1}, w) \leq C(C\varepsilon^{1+\frac{k\alpha}{M}})^{1+\frac{\alpha}{M}} \leq \varepsilon^{1+\frac{k+1}{M}\alpha}$ .

A curve  $\Gamma_m = \gamma_1 \cup \dots \cup \gamma_m$  is horizontal, has endpoints  $v$  and  $w_m$ , its length does not exceed  $l(\gamma_1) + C \sum_{l=1}^m \varepsilon^{1+\frac{l\alpha}{M}} \leq l(\gamma_1) + C\varepsilon^{1+\frac{\alpha}{M}}$  and  $d_\infty(w_m, w) \rightarrow 0$  as

$m \rightarrow \infty$ . Therefore the sequence  $\Gamma_m$  converges to a horizontal curve  $\gamma$  as  $m \rightarrow \infty$  with properties 1–2 mentioned in the theorem.

Under  $v = g$  we can take  $d_\infty(g, w)$  as  $\varepsilon$  in above estimates: it gives an evaluation  $l(\gamma) \leq C d_\infty(g, w)$ . The opposite inequality can be verified directly by means of the above obtained estimate: indeed, if  $d_\infty(g, w) = \varepsilon$  then  $d_\infty(g, w) = d_1^g(g, w) \leq C \widehat{l}(\widehat{\gamma}) \leq C l(\gamma) + o(\widehat{l}(\widehat{\gamma}))$ ; it follows that  $d_\infty(g, w) - o(d_\infty(g, w)) \leq C l(\gamma)$  and the estimate  $d_\infty(g, w) \leq C_1 l(\gamma)$  holds with  $C_1$  independent of  $g$  from some compact neighborhood if  $v$  is close enough to  $g$ . Thus we have obtained the property 3.  $\square$

As an application of Theorem 2.8.4 we obtain a version of Rashevskii–Chow type connectivity theorem.

**Theorem 2.8.5.** *Suppose that Assumption 2.1.4 holds for  $\alpha \in (0, 1]$ . Every two points  $v, w$  of a connected Carnot manifold can be joined by a rectifiable absolutely continuous horizontal curve  $\gamma$  composed by not more than countably many segments of integral lines of given horizontal fields.*

## 2.8.2 Comparison of metrics, and Ball–Box Theorem

**Corollary 2.8.6.** *Suppose that Assumption 2.1.4 holds for  $\alpha \in (0, 1]$ . In some compact neighborhood the distance  $d_c$  is equivalent to the quasimetric  $d_\infty$ .*

*Proof.* An estimate  $d_c(x, y) \leq C_1 d_\infty(x, y)$  for points  $x, y$  from a compact part  $\mathbb{M}$  follows from Theorem 2.8.4. Our next goal is to prove the converse estimate. Fix a compact part  $K \subset \mathbb{M}$  and assume the contrary: for any  $n \in \mathbb{N}$  there exist points  $x_n, y_n \in K$  such that  $d_\infty(x_n, y_n) \geq n d_c(x_n, y_n)$ . In this case we have  $d_\infty(x_n, y_n) \rightarrow 0$  as  $n \rightarrow \infty$  since otherwise we have simultaneously  $d_c(x_n, y_n) \rightarrow 0$  as  $n \rightarrow \infty$ , and  $d_\infty(x_n, y_n) \geq \alpha > 0$  for all  $n \in \mathbb{N}$  what is impossible. We can assume also that  $x_n \rightarrow x \in K$  as  $n \rightarrow \infty$  and  $x_n \neq y_n$ . Setting  $d_\infty(x_n, y_n) = \varepsilon_n$  we have  $d_\infty(x_n, \Delta_{r_0 \varepsilon_n}^{x_n} y_n) = r_0$ , and  $d_c^n(x_n, \Delta_{r_0 \varepsilon_n}^{x_n} y_n) \leq r_0 n^{-1}$ , where the distance  $d_c^n$  is measured with respect to the frame  $\{X_i^{\varepsilon_n}\}$  with pushed-forward Riemannian tensor. As far as the length of vectors  $X_i^{\varepsilon_n}$ ,  $i = 1, \dots, \dim H_1$ , is closed to the lengths of corresponding nilpotentized vector fields  $\widehat{X}_i$ ,  $i = 1, \dots, \dim H_1$ , by Corollary 2.2.11, the Riemannian distance  $\rho(x_n, \Delta_{r_0 \varepsilon_n}^{x_n} y_n) \rightarrow 0$  as  $n \rightarrow \infty$ . It is in a contradiction with  $d_\infty(x_n, \Delta_{r_0 \varepsilon_n}^{x_n} y_n) = r_0$  for all  $n \in \mathbb{N}$  (see Proposition 2.8.12 for a comparison of metrics).  $\square$

**Remark 2.8.7.** Note that, for obtaining the estimate  $d_\infty(x, y) \leq C_2 d_c(x, y)$ , the value  $\alpha$  need not to be strictly greater than zero. Thus, the estimate  $d_\infty(x, y) \leq C_2 d_c(x, y)$  is valid also for  $\alpha = 0$ .

Another corollary is so called ball-box theorem proved for smooth vector fields in [110, 68].

**Theorem 2.8.8** (Ball–Box Theorem). *Suppose that Assumption 2.1.4 holds for  $\alpha \in (0, 1]$ . The shape of a small ball  $B(x, r)$  in the metric  $d_c$  looks like a box: given compact set  $K \subset \mathbb{M}$  there are constants  $0 < C_1 \leq C_2 < \infty$  and  $r_0$  independent from  $x \in K$  such that*

$$\text{Box}(x, C_1 r) \subset B(x, r) \subset \text{Box}(x, C_2 r) \quad (2.8.3)$$

for all  $r \in (0, r_0)$ .

Theorem 2.8.8 implies

**Corollary 2.8.9.** *Suppose that Assumption 2.1.4 holds for  $\alpha \in (0, 1]$ . The Hausdorff dimension of  $\mathbb{M}$  equals*

$$\nu = \sum_{i=1}^M i(\dim H_i - \dim H_{i-1})$$

where  $\dim H_0 = 0$ .

This Corollary extends Mitchell Theorem [102] to Carnot–Carathéodory spaces with minimal smoothness of vector fields.

**Remark 2.8.10.** Let Assumption 2.1.4 holds for  $\alpha \in (0, 1]$ . Applying Corollary 2.8.8, we obtain

1. the generalization of Theorem 2.3.9 for points  $w$  and  $w'$  close enough:

$$\max\{d_c^u(w, w'), d_c(w, w')\} = \Theta[\rho(u, v)\rho(v, w)]^{\frac{1}{M}} \leq \Theta[d_c(u, v)d_c(v, w)]^{\frac{1}{M}};$$

2. the generalization of Theorem 2.4.1:

$$\max\{d_c^u(w_\varepsilon, w'_\varepsilon), d_c(w_\varepsilon, w'_\varepsilon)\} = \varepsilon[\Theta(C, \mathcal{C})]\rho(u, v)^{\frac{\alpha}{M}}(d_c^u(v, w) + o(1));$$

3. the generalization of Theorem 2.7.1:

$$\max\{d_c^u(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon), d_c(\widehat{w}_Q^\varepsilon, w_Q^\varepsilon)\} = \sum_{k=1}^Q \Theta(\mathcal{C}, k, \{F_{\mu, \beta}^j\}_{j, \mu, \beta}) \cdot \varepsilon^{1 + \frac{\alpha}{M}}.$$

Corollary 2.8.6 and Theorem 11.11 [70] imply the following statement containing a result of [66] where only the first assertion is obtained under assumption of higher smoothness of vector fields.

**Proposition 2.8.11.** *Let  $X$  and  $Y$  be two families of vector fields on  $\mathbb{M}$  with the same horizontal distribution  $H\mathbb{M}$  for both of which Assumption 2.1.4 holds with some  $\alpha \in (0, 1]$ . Then in some compact neighborhood the following assertions are equivalent:*

- 1) *There exists a constant  $C \geq 1$  such that  $C^{-1}d_\infty^X \leq d_\infty^Y \leq Cd_\infty^X$ .*
- 2) *There exists a constant  $C \geq 1$  such that  $C^{-1}|X_H\varphi| \leq |Y_H\varphi| \leq C|X_H\varphi|$  for all  $\varphi \in C^\infty(\mathbb{M})$ .*

Here  $d_\infty^X$  and  $d_\infty^Y$  are quasimetrics constructed with respect to the bases  $X$  and  $Y$ , and  $X_H\varphi$  and  $Y_H\varphi$  are subgradients of  $\varphi$ .

Define the Riemannian quasimetric  $d_{\text{riem}}(u, v)$  between a point  $u$  and a point  $v = \exp\left(\sum_{i=1}^N x_i X_i\right)(u)$  as  $d_{\text{riem}}(u, v) = \max\{|x_i| \mid i = 1, \dots, N\}$ . The well-known facts of differential geometry imply that the metric  $d_{\text{riem}}$  is equivalent to the Riemannian metric  $\rho$  on every compactly embedded domain  $U \subseteq \mathbb{M}$ , i.e., there exists a constant  $c$  independent of the choice of the points  $u, v \in U$  and such that  $c^{-1}\rho(u, v) \leq d_{\text{riem}}(u, v) \leq c\rho(u, v)$  for all  $u, v \in U$  for which the quantities under consideration are defined. Hence, we have

**Proposition 2.8.12.** *The relations*

$$c^{-1}\rho(u, v) \leq d_{\text{riem}}(u, v) \leq cd_\infty(u, v) \leq cd_{\text{riem}}(u, v)^{\frac{1}{M}}$$

*hold for all  $x, y \in U$ .*

**Remark 2.8.13.** If the derivatives of  $X_i$ ,  $i = 1, \dots, N$ , are locally Hölder with respect to  $\mathfrak{d}$ , where  $\mathfrak{d}$  meets conditions of Remark 2.7.2, the statements of Corollary 2.8.3, Theorem 2.8.4, Theorem 2.8.5, Corollary 2.8.6, Theorem 2.8.8, Corollary 2.8.9, Remark 2.8.10 and Proposition 2.8.11 are also true.

## 3 Differentiability on a Carnot Manifold

### 3.1 Primitive calculus

Further, we extend the dilations  $\delta_t^g$  to negative  $t$  by setting  $\delta_t^g x = \delta_{|t|}^g(x^{-1})$  for  $t < 0$ . The convenience of this definition is seen from the comparison of different kinds of differentiability.

### 3.1.1 Definition

Let  $\mathbb{M}, \mathbb{N}$  be two Carnot manifolds. We denote the vector fields on  $\mathbb{N}$  by  $Y_i$ . We label the remaining objects on  $\mathbb{N}$  (the distance, the tangent cone etc.) with the same symbols as on  $\mathbb{M}$  but with a tilde  $\sim$  excluding the cases where the objects under consideration are obvious: for example, for a given mapping  $f : E \rightarrow \mathbb{N}$ , it is clear that  $\mathcal{G}^g\mathbb{M}$  is the tangent cone at a point  $g \in \mathbb{M}$  and  $\mathcal{G}^{f(g)}\mathbb{N}$  is the tangent cone at the point  $f(g) \in \mathbb{N}$ ;  $d_c^g$  is the metric in the cone  $\mathcal{G}^g\mathbb{M}$ ,  $d_c^{f(g)}$  is the metric in  $\mathcal{G}^{f(g)}\mathbb{N}$ , etc.

Recall that a *horizontal homomorphism* of Carnot groups is a continuous homomorphism  $L : \mathbb{G} \rightarrow \tilde{\mathbb{G}}$  of Carnot groups such that

$$1) DL(0)(H\mathbb{G}) \subset H\tilde{\mathbb{G}}.$$

The notion of a horizontal homomorphism  $L : (\mathcal{G}^g\mathbb{M}, d_c^g) \rightarrow (\mathcal{G}^q\mathbb{N}, \tilde{d}_c^q)$ ,  $g \in \mathbb{M}$ ,  $q \in \mathbb{N}$ , of local Carnot groups is different from this only in that the inclusion  $L(\mathcal{G}^g\mathbb{M} \cap \exp H\mathcal{G}^g\mathbb{M}) \subset \mathcal{G}^q\mathbb{N} \cap \exp H\mathcal{G}^q\mathbb{N}$  holds only for  $v \in \mathcal{G}^g\mathbb{M} \cap \exp H\mathcal{G}^g\mathbb{M}$  such that  $L(v) \in \mathcal{G}^q\mathbb{N}$ .

Since a homomorphism of Lie groups is continuous, it can be proved that a horizontal homomorphism  $L : \mathbb{G} \rightarrow \tilde{\mathbb{G}}$  also has the property

2)  $L(\delta_t v) = \tilde{\delta}_t L(v)$  for all  $v \in \mathbb{G}$  and  $t > 0$  (in the case of a horizontal homomorphism  $L : (\mathcal{G}^g\mathbb{M}, d_c^g) \rightarrow (\mathcal{G}^q\mathbb{N}, \tilde{d}_c^q)$  of local Carnot groups, the equality  $L(\delta_t v) = \tilde{\delta}_t L(v)$  is fulfilled only for  $v \in \mathcal{G}^g\mathbb{M}$  and  $t > 0$  such that  $\delta_t v \in \mathcal{G}^g\mathbb{M}$  and  $\tilde{\delta}_t L(v) \in \mathcal{G}^q\mathbb{N}$ ).

**Definition 3.1.1.** Given two Carnot manifolds  $\mathbb{M}$  and  $\mathbb{N}$ , and a set  $E \subset \mathbb{M}$ , a mapping  $f : E \rightarrow \mathbb{N}$  is called *hc-differentiable* at a point  $g \in E$  if there exists a horizontal homomorphism  $L : (\mathcal{G}^g\mathbb{M}, d_c^g) \rightarrow (\mathcal{G}^{f(g)}\mathbb{N}, d_c^{f(g)})$  of the nilpotent tangent cones such that

$$d_c^{f(g)}(f(v), L(v)) = o(d_c^g(g, v)) \quad \text{as } E \cap \mathcal{G}^g\mathbb{M} \ni v \rightarrow g. \quad (3.1.1)$$

A horizontal homomorphism  $L : (\mathcal{G}^g\mathbb{M}, d_c^g) \rightarrow (\mathcal{G}^{f(g)}\mathbb{N}, d_c^{f(g)})$  satisfying condition (3.1.1), is called a *hc-differential* of the mapping  $f : E \rightarrow \mathbb{N}$  at  $g \in E$  on  $E$  and is denoted by  $Df(g)$ . It can be proved [129] that if  $g$  is a density point then the *hc-differential* is unique.

Moreover, it is easy to verify that the *hc-differential* commutes with the one-parameter dilation group:

$$\delta_t^{f(g)} \circ Df(g) = Df(g) \circ \delta_t^g. \quad (3.1.2)$$

**Proposition 3.1.2** ([129]). *Definition 3.1.1 is equivalent to each of the following assertion:*



1.  $d_c^{f(g)}(\Delta_{t^{-1}}^{f(g)} f(\delta_t^g(v)), L(v)) = o(1)$  as  $t \rightarrow 0$ , where  $o(\cdot)$  is uniform in the points  $v$  of any compact part of  $\mathcal{G}^g \mathbb{M}$ ;
2.  $\tilde{d}_\infty(f(v), L(v)) = o(d_c^g(g, v))$  as  $E \cap \mathcal{G}^g \mathbb{M} \ni v \rightarrow g$ ;
3.  $\tilde{d}_\infty(f(v), L(v)) = o(d_\infty(g, v))$  as  $E \cap \mathcal{G}^g \mathbb{M} \ni v \rightarrow g$ ;
4.  $\tilde{d}_\infty(\Delta_{t^{-1}}^{f(g)} f(\delta_t^g(v)), L(v)) = o(1)$  as  $t \rightarrow 0$ , where  $o(\cdot)$  is uniform in the points  $v$  of any compact part of  $\mathcal{G}^g \mathbb{M}$ .

*Proof.* Consider a point  $v$  of a compact part of  $\mathcal{G}^g \mathbb{M}$  and a sequence  $\varepsilon_i \rightarrow 0$  as  $i \rightarrow \infty$  such that  $\delta_{\varepsilon_i}^g v \in E$  for all  $i \in \mathbb{N}$ . From (3.1.1) we have  $d_c^{f(g)}(f(\delta_{\varepsilon_i}^g v), L(\delta_{\varepsilon_i}^g v)) = o(d_c^g(g, \delta_{\varepsilon_i}^g v)) = o(\varepsilon_i)$ . In view of (3.1.2), we infer

$$d_c^{f(g)}(\Delta_{\varepsilon_i}^{f(g)}(\Delta_{\varepsilon_i^{-1}}^{f(g)} f(\delta_{\varepsilon_i}^g v)), \delta_{\varepsilon_i}^{f(g)} L(v)) = o(\varepsilon_i) \quad \text{uniformly in } v.$$

From here we obtain item 1. Obviously, the argument is reversible. Item 1 is equivalent to item 4 since  $\alpha \rho(x, y) \leq d_\infty(x, y) \leq \beta \rho(x, y)^{\frac{1}{M}}$  ( $\alpha \rho(x, y) \leq d_\infty^g(x, y) \leq \beta \rho(x, y)^{\frac{1}{M}}$ ) on any compact part of  $\mathbb{M} \cap \mathcal{G}^g \mathbb{M}$  ( $\alpha$  and  $\beta$  depend on the choice of the compact part).

By comparing the metrics of Subsection 2.8:  $d_\infty^{f(g)}(g, v) = O(d_c^{f(g)}(g, v))$ , and by Local Approximation Theorem 2.5.4, we obtain the equivalence of (3.1.1) to the item 2. The equivalence of items 2 and 3 is obtained by comparing the metrics of Subsection 2.8:  $d_c^g(g, v) = O(d_\infty^g(g, v))$  and  $d_\infty^g(g, v) = O(d_\infty(g, v))$  as  $v \rightarrow g$ .  $\square$

### 3.1.2 Chain rule

In this subsection, we prove the chain rule.

**Theorem 3.1.3** ([129]). *Suppose that  $\mathbb{M}, \mathbb{N}, \mathbb{X}$  are Carnot manifolds,  $E$  is a set in  $\mathbb{M}$ , and  $f : E \rightarrow \mathbb{N}$  is a mapping from  $E$  into  $\mathbb{N}$   $hc$ -differentiable at a point  $g \in E$ . Suppose also that  $F$  is a set in  $\mathbb{N}$ ,  $f(E) \subset F$  and  $\varphi : F \rightarrow \mathbb{X}$  is a mapping from  $F$  into  $\mathbb{X}$   $hc$ -differentiable at  $p = f(g) \in \mathbb{N}$ . Then the composition  $\varphi \circ f : E \rightarrow \mathbb{X}$  is  $hc$ -differentiable at  $g$  and*

$$D(\varphi \circ f)(g) = D\varphi(p) \circ Df(g).$$

*Proof.* By hypothesis,  $d_c^{f(g)}(f(v), Df(g)(v)) = o(d_c^g(g, v))$  as  $v \rightarrow g$  and also

$d_c^{\varphi(p)}(\varphi(w), D\varphi(p)(w)) = o(d_c^p(p, w))$  as  $w \rightarrow p$ . We now infer

$$\begin{aligned} & d_c^{\varphi(p)}((\varphi \circ f)(v), (D\varphi(p) \circ Df(g))(v)) \\ & \leq d_c^{\varphi(p)}(\varphi(f(v)), D\varphi(p)(f(v))) + d_c^{\varphi(p)}(D\varphi(p)(f(v)), D\varphi(p)(Df(g)(v))) \\ & \leq o(d_c^p(p, f(v))) + O(d_c^p(f(v), Df(g)(v))) \\ & \leq o(d_c^g(g, v)) + O(o(d_c^g(g, v))) = o(d_c^g(g, v)) \quad \text{as } v \rightarrow g, \end{aligned}$$

since

$$\begin{aligned} d_c^p(p, f(v)) & \leq d_c^p(p, Df(g)(v)) + d_c^p(f(v), Df(g)(v)) \\ & = O(d_c^g(g, v)) + o(d_c^g(g, v)) = O(d_c^g(g, v)) \quad \text{as } v \rightarrow g. \end{aligned}$$

(The estimate  $d_c^p(p, Df(g)(v)) = O(d_c^g(g, v))$  as  $v \rightarrow g$  follows from the continuity of the homomorphism  $Df(g)$  and (3.1.2).)  $\square$

## 3.2 $hc$ -Differentiability of Curves on the Carnot Manifolds

### 3.2.1 Coordinate $hc$ -differentiability criterion

Recall that a mapping  $\gamma : E \rightarrow \mathbb{M}$ , where  $E \subset \mathbb{R}$  is an arbitrary set, is called a *Lipschitz mapping* if there exists a constant  $L$  such that the inequality  $d_\infty(\gamma(y), \gamma(x)) \leq L|y - x|$  holds for all  $x, y \in E$ .

**Definition 3.2.1.** A mapping  $\gamma : E \rightarrow \mathbb{M}$ , where  $E \subset \mathbb{R}$  is an arbitrary set, is called *hc-differentiable at a limit point*  $s \in E$  to  $E$  if there exists a horizontal vector  $a = \sum_{i=1}^{\dim H_1} \alpha_i \widehat{X}_i^{\gamma(s)}(\gamma(s)) \in H_{\gamma(s)}\mathbb{M}$  such that the local homomorphism  $\tau \mapsto \exp\left(\tau \sum_{i=1}^{\dim H_1} \alpha_i \widehat{X}_i^{\gamma(s)}\right)(\gamma(s)) \in \mathcal{G}^{\gamma(s)}\mathbb{M}$  as the  $hc$ -differential of the mapping  $\gamma : E \rightarrow \mathbb{M}$ :  $d_c^{\gamma(s)}(\gamma(s+\tau), \delta_\tau^{\gamma(s)}a) = o(\tau)$  for  $\tau \rightarrow 0$ ,  $s+\tau \in E$ . The point  $\exp\left(\sum_{i=1}^{\dim H_1} \alpha_i \widehat{X}_i^{\gamma(s)}\right)(\gamma(s)) \in \mathcal{G}^{\gamma(s)}\mathbb{M}$  is called the *hc-derivative*<sup>2</sup>.

Some properties of the introduced notion of  $hc$ -differentiability can be obtained from Proposition 3.1.2. For instance, the coefficients  $\alpha_i$  are defined uniquely: if, in the normal coordinates,  $\gamma(s+\tau) = \exp\left(\sum_{i=1}^N \gamma_i(\tau) \widehat{X}_i^{\gamma(s)}\right)(\gamma(s))$ ,  $s+\tau \in E$ , for sufficiently small  $\tau$  then Proposition 3.1.2 implies:

---

<sup>2</sup>If the  $hc$ -derivative does not exist in  $\mathcal{G}^{\gamma(s)}\mathbb{M}$  then it belongs in  $\mathbb{G}_{\gamma(s)}\mathbb{M}$ : we consider the “preimage” under  $\theta_{\gamma(s)}$  being equal  $\exp\left(\sum_{i=1}^{\dim H_1} \alpha_i (\widehat{X}_i^{\gamma(s)})'\right)(0)$  in all the necessary cases.

**Property 3.2.2** ([129]). A mapping  $\gamma : [a, b] \rightarrow \mathbb{M}$  is *hc*-differentiable at a point  $s \in (a, b)$  if and only if one of the following assertions holds:

(1)  $\gamma_i(\tau) = \alpha_i \tau + o(\tau)$ ,  $i = 1, \dots, \dim H_1$ , and  $\gamma_i(\tau) = o(\tau^{\deg X_i})$ ,  $i > \dim H_1$ , as  $\tau \rightarrow 0$ ,  $s + \tau \in E$ ;

(2) the vector  $\sum_{i=1}^{\dim H_1} \alpha_i \widehat{X}_i^{\gamma(s)}(\gamma(s)) \in H_{\gamma(s)}\mathbb{M}$  is the Riemannian derivative of  $\gamma : [a, b] \rightarrow \mathbb{M}$  at a point  $s \in (a, b)$ , and  $\gamma_i(\tau) = o(\tau^{\deg X_i})$ ,  $i > \dim H_1$ , as  $\tau \rightarrow 0$ ,  $s + \tau \in E$ .

### 3.2.2 *hc*-Differentiability of absolutely continuous curves

If a curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  is absolutely continuous in the Riemannian sense then all coordinate functions  $\gamma_i(t)$  are absolutely continuous on the closed interval  $[a, b]$  (it is clear that this property is independent of the choice of the coordinate system). Therefore the tangent vector  $\dot{\gamma}(t)$  is defined almost everywhere on  $[a, b]$ . If, moreover,  $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$  at the points  $t \in [a, b]$  of Riemannian differentiability then the curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  is called *horizontal*.

It is well known that almost all points  $t$  of a closed interval  $E = [a, b]$  are Lebesgue points of the derivatives of the horizontal components, that is, if, in the normal coordinates  $\gamma(t + \tau) = \exp\left(\sum_{j=1}^N \gamma_j(\tau) X_j\right)(\gamma(t))$  then the *horizontal* components  $\gamma_j(\sigma)$ ,  $j = 1, \dots, \dim H_1$ , have the property

$$\int_{\{\sigma \in (\alpha, \beta) \mid t + \sigma \in E\}} |\dot{\gamma}_j(\sigma) - \dot{\gamma}_j(0)| d\sigma = o(\beta - \alpha) \quad \text{as } \beta - \alpha \rightarrow 0 \quad (3.2.1)$$

on intervals  $(\alpha, \beta) \ni 0$ . Note that property 3.2.1 is independent of the choice of the coordinate system in a neighborhood of  $\gamma(t)$ .

**Theorem 3.2.3** ([129]). *Let a curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  on a Carnot manifold be absolutely continuous in the Riemannian sense and horizontal. Then  $\gamma : [a, b] \rightarrow \mathbb{M}$  is *hc*-differentiable almost everywhere: any point  $t \in [a, b]$  which is a Lebesgue point of the derivatives of its horizontal components is also a point at which  $\gamma$  is *hc*-differentiable. If  $\gamma(t + \tau) = \exp\left(\sum_{j=1}^N \gamma_j(\tau) X_j\right)(\gamma(t))$  then *hc*-derivative  $\dot{\gamma}(t)$  equals*

$$\exp\left(\sum_{j=1}^{\dim H_1} \dot{\gamma}_j(0) \widehat{X}_j^{\gamma(t)}\right)(\gamma(t)) = \exp\left(\sum_{j=1}^{\dim H_1} \dot{\gamma}_j(0) X_j\right)(\gamma(t)).$$

*Proof.* Fix a Lebesgue point  $t_0 \in (a, b)$  of the derivatives of the horizontal components of the mapping  $\gamma(t_0 + \tau) = \exp\left(\sum_{j=1}^N \gamma_j(\tau) X_j\right)(g)$ ,  $g = \gamma(t_0)$ . In this proof, we also fix a normal coordinate system  $\theta_g$  at  $g$ . To simplify the notation, we write the vector fields  $\tilde{X}_i^g = (\theta_g^{-1})_* X_i$  and  $\hat{X}_i'^g = (\theta_g^{-1})_* \hat{X}_i^g$  defined in a neighborhood of  $0 \in \mathbb{R}^N$  without the superscript  $g$ :  $\tilde{X}_i = (\theta_g^{-1})_* X_i$  and  $\hat{X}_i' = (\theta_g^{-1})_* \hat{X}_i^g$  respectively.

For proving the  $hc$ -differentiability of the mapping  $\gamma$  at  $t_0$ , we need to establish the estimate  $\gamma_j(\tau) = o(\tau^{\deg X_j})$  as  $\tau \rightarrow 0$  for all  $j > \dim H_1$ ,  $t_0 + \tau \in [a, b]$  (see Property 3.2.2). Partition the proof of the desired estimate into several steps.

**1<sup>ST</sup> STEP.** Here we show that the hypothesis implies the Riemannian differentiability of the mapping  $\gamma$  at  $t_0$  and  $\dot{\gamma}(t_0) \in H_g \mathbb{M}$ . Put  $\Gamma(t_0 + \tau) = \theta_g^{-1}(\gamma(t)) = (\gamma_1(\tau), \dots, \gamma_N(\tau))$ . The curve  $\Gamma(\tau)$  is absolutely continuous, and its tangent vector  $\dot{\Gamma}(\tau)$  is horizontal in a neighborhood of  $0 \in T_g \mathbb{M}$  with respect to the vector fields  $\{\tilde{X}_i\}$ :  $\dot{\Gamma}(\tau) \in (\theta_g^{-1})_* H_{\gamma(t_0 + \tau)} \mathbb{M}$  for almost all  $\tau$ . From here, for almost all  $\tau$  sufficiently closed to 0, we infer

$$\dot{\Gamma}(\tau) = \sum_{j=1}^N \dot{\gamma}_j(\tau) \frac{\partial}{\partial x_j} = \sum_{i=1}^{\dim H_1} a_i(\tau) \tilde{X}_i(\Gamma(\tau)). \quad (3.2.2)$$

The Riemann tensor pulled back from the manifold  $\mathbb{M}$  onto a neighborhood of  $0 \in T_g \mathbb{M}$  is continuous at the zero. Therefore, using this continuity, we see that, for any  $\tau$ ,  $t_0 + \tau \in [a, b]$ , (3.2.1) implies

$$\begin{aligned} d_c(\gamma(t_0), \gamma(t_0 + \tau)) &\leq c_1 \int_{(0, \tau)} |\dot{\Gamma}(\sigma)|_r d\sigma \\ &\leq c_2 \sum_{j=1}^{\dim H_1} \int_{(0, \tau)} (|\dot{\gamma}_j(\sigma) - \dot{\gamma}_j(0)| + |\dot{\gamma}_j(0)|) d\sigma = O(\tau) \end{aligned}$$

as  $\tau \rightarrow 0$ , where  $|\dot{\Gamma}(\sigma)|_r$  stands for the length of the tangent vector in the pulled-back Riemannian metric. By Proposition 2.8.6 and Remark 2.8.7, we have  $d_\infty(\gamma(t_0), \gamma(t_0 + \tau)) = O(d_c(\gamma(t_0), \gamma(t_0 + \tau)))$  as  $\tau \rightarrow 0$ . Therefore the coordinate components  $\gamma_j(\tau)$  of the mapping  $\gamma$  satisfy

$$\gamma_j(\tau) = O(\tau^{\deg X_j}) \quad \text{as } \tau \rightarrow 0 \text{ for all } j \geq 1. \quad (3.2.3)$$

It follows that the curve  $\Gamma(\tau)$  is differentiable at 0 and

$$\dot{\Gamma}(0) = (\dot{\gamma}_1(0), \dots, \dot{\gamma}_{\dim H_1}(0), 0, \dots, 0).$$

Hence, the curve  $\gamma$  is differentiable in the Riemannian sense at  $t_0$  and  $\dot{\gamma}(t_0) \in H_g \mathbb{M}$ . From (3.2.3) we also obtain  $\gamma(\tau) \in B(g, O(\tau))$ .

2<sup>ND</sup> STEP. Corollary 2.2.9 and the fact that  $\gamma(\tau) \in B(g, O(\tau))$  imply that, in a neighborhood of 0, the vector fields  $\tilde{X}_i$  can be expressed via  $\hat{X}'_k$  so that

$$\tilde{X}_i(\Gamma(\tau)) = \sum_{k=1}^N \alpha_{ik}(\tau) \hat{X}'_k(\Gamma(\tau)), \text{ where } \alpha_{ik}(\tau) = \begin{cases} o(\tau^{\deg X_k - \deg X_i}) & \text{if} \\ \deg X_k > \deg X_i, \\ \delta_{ik} + o(1) & \text{otherwise} \end{cases}$$

as  $\tau \rightarrow 0$ . Now, using expansion (2.1.5) of the vector fields  $\hat{X}'_i$  in the standard Euclidean basis, for all points  $\tau$  sufficiently close to 0, from (3.2.2) we now obtain

$$\begin{aligned} \sum_{j=1}^N \dot{\gamma}_j(\tau) \frac{\partial}{\partial x_j} &= \sum_{i=1}^{\dim H_1} a_i(\tau) \tilde{X}_i(\Gamma(\tau)) = \sum_{k=1}^N \sum_{i=1}^{\dim H_1} a_i(\tau) \alpha_{ik}(\tau) \hat{X}'_k(\Gamma(\tau)) \\ &= \sum_{j=1}^N \sum_{k=1}^j \sum_{i=1}^{\dim H_1} a_i(\tau) \alpha_{ik}(\tau) \hat{z}_k^j(\Gamma(\tau)) \frac{\partial}{\partial x_j}. \end{aligned} \quad (3.2.4)$$

3<sup>RD</sup> STEP. For  $1 \leq j \leq \dim H_1$ , we have  $\deg X_j = 1$ . Then from (3.2.3) and (2.1.5) we conclude that  $\hat{z}_k^j(\Gamma(\tau)) = \delta_{jk} + O(\tau)$ . Therefore, from (3.2.4)

we infer  $\dot{\gamma}_j(\tau) = \sum_{i=1}^{\dim H_1} a_i(\tau) \tilde{\alpha}_{ij}(\tau)$ , where, as before,  $\tilde{\alpha}_{ij}(\tau) = \delta_{ij} + o(1)$ .

Hence,  $a_i(\tau) = \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(\tau) \beta_{ni}(\tau)$ , where  $\{\beta_{ni}(\tau)\}$ ,  $n, i = 1, \dots, \dim H_1$ , is a matrix inverse to  $\{\tilde{\alpha}_{ij}(\tau)\}$ , has the elements  $\beta_{ni}(\tau) = \delta_{ni} + o(1)$ . Consequently,

$$\begin{aligned} a_i(\tau) &= \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(\tau) \beta_{ni}(\tau) = \sum_{n=1}^{\dim H_1} \dot{r}_n(\tau) \beta_{ni}(\tau) + \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(0) \beta_{ni}(\tau), \\ &\text{where } r_n(\tau) = \int_0^\tau (\dot{\gamma}_n(\sigma) - \dot{\gamma}_n(0)) d\sigma. \end{aligned} \quad (3.2.5)$$

4<sup>TH</sup> STEP. Fix  $\dim H_{l-1} < j \leq \dim H_l$ ,  $1 < l \leq M$ . For estimating  $\dot{\gamma}_j(\tau)$ ,

from (3.2.4) we have

$$\begin{aligned}\dot{\gamma}_j(\tau) &= \sum_{k,i,n=1}^{\dim H_1} \dot{\gamma}_n(\tau) \beta_{ni}(\tau) \alpha_{ik}(\tau) \hat{z}_k^j(\Gamma(\tau)) \\ &\quad + \sum_{k=\dim H_1+1}^j \sum_{i,n=1}^{\dim H_1} \dot{\gamma}_n(\tau) \beta_{ni}(\tau) \alpha_{ik}(\tau) \hat{z}_k^j(\Gamma(\tau)) = I_j + II_j. \quad (3.2.6)\end{aligned}$$

Since in this case  $\alpha_{ik}(\tau) = o(\tau^{\deg X_k - \deg X_i})$ , and  $\hat{z}_k^j(\Gamma(\tau)) = O(\tau^{\deg X_j - \deg X_k})$  by (3.2.3) then all components in the double sum in (3.2.6) have a factor  $o(\tau^{l-1})$ . Therefore

$$II_j = \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(\tau) o(\tau^{l-1}). \quad (3.2.7)$$

From another side

$$\begin{aligned}I_j &= \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(\tau) \hat{z}_n^j(\Gamma(\tau)) + \sum_{k,n=1}^{\dim H_1} \dot{\gamma}_n(\tau) o(1) \hat{z}_k^j(\Gamma(\tau)) \\ &= \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(0) \hat{z}_n^j(\Gamma(\tau)) + \sum_{n=1}^{\dim H_1} \dot{r}_n(\tau) \hat{z}_n^j(\Gamma(\tau)) + \sum_{k,n=1}^{\dim H_1} \dot{\gamma}_n(\tau) o(1) \hat{z}_k^j(\Gamma(\tau)) \\ &= \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(0) \sum_{|\alpha+e_n|_h=\deg X_j, \alpha>0} F_{\alpha,e_n}^j(g) \Gamma(\tau)^\alpha \\ &\quad + \sum_{n=1}^{\dim H_1} \dot{r}_n(\tau) O(\tau^{l-1}) + \sum_{n=1}^{\dim H_1} \dot{\gamma}_n(\tau) o(\tau^{l-1}). \quad (3.2.8)\end{aligned}$$

In the estimation of the increment of  $\gamma_j(\tau)$  on  $[0, \tau]$  by the Newton–Leibnitz formula, the components of (3.2.7) and the last two summands in (3.2.8) have order  $o(\tau^l)$ . Indeed, for all  $1 \leq i \leq \dim H_1$  and  $s > 0$ , from (3.2.1) and (3.2.5) we have  $|\dot{\gamma}_n(\tau)| \leq |\dot{\gamma}_n(0)| + |\dot{r}_i(\tau)|$  from (3.2.5),  $|r_i(\tau)| \leq \int_0^\tau |\dot{\gamma}_i(\sigma) - \dot{\gamma}_i(0)| d\sigma = o(\tau)$  and

$$\left| \int_0^\tau \dot{r}_i(\sigma) O(\sigma^s) d\sigma \right| \leq |O(\tau^s)| \int_0^\tau |\dot{\gamma}_i(\sigma) - \dot{\gamma}_i(0)| d\sigma = o(\tau^{s+1}).$$

**5<sup>TH</sup> STEP.** In the remaining double sum in (3.2.8), the summands with index  $\alpha$  for which  $|\alpha + e_n| < \deg X_j$  contain the factor  $\Gamma(\tau)^\alpha = o(\tau^{l-1})$ ,

since, in this case, the product  $\Gamma(\tau)^\alpha$  necessarily contains the factor  $\gamma_j(\tau) = \dot{\gamma}_j(0)\tau + o(\tau) = o(\tau)$ ,  $j > \dim H_1$ . Therefore, expression (3.2.8) for  $\dot{\gamma}_j(\tau)$  is reduced to the following:

$$\dot{\gamma}_j(\tau) = \sum_{i=1}^{\dim H_1} \dot{\gamma}_i(0) \sum_{\substack{|\alpha+e_n|_h=\deg X_j, \\ |\alpha+e_n|=\deg X_j}} F_{\alpha,e_n}^j(g) \Gamma(\tau)^\alpha + o(\tau^{l-1}). \quad (3.2.9)$$

Since also  $\Gamma(\tau) = \dot{\Gamma}(0)\tau + o(\tau)$ , we see that each summand in (3.2.9) is equal to  $\dot{\gamma}_i(0)F_{\alpha,e_n}^j(g)\Gamma(\tau)^\alpha = \tau^{l-1}\dot{\gamma}_i(0)F_{\alpha,e_n}^j(g)\dot{\Gamma}(0)^\alpha + o(\tau^{l-1})$ . Consequently, leaving only the summands of order  $\tau^{l-1}$  in (3.2.9), we have

$$\dot{\gamma}_j(\tau) = \sum_{i=1}^{\dim H_1} \tau^{l-1} \sum_{|\alpha|=|\alpha|_h=l-1} \dot{\gamma}_i(0)F_{\alpha,e_n}^j(g)\dot{\Gamma}(0)^\alpha + o(\tau^{l-1}). \quad (3.2.10)$$

Similarly, the second summand in the estimation of the increment  $\gamma_j(\tau)$  is equal to  $o(\tau^l)$ . Consequently, for the validity of the theorem, it is necessary and sufficient that the double sum in (3.2.10) be zero. This was established in Lemma 2.1.21.

Thus, we have proved that  $\gamma_j(\tau) = o(\tau^{\deg X_j})$  for all  $j > \dim H_1$ . Since the horizontal components of  $\gamma$  are differentiable at  $t_0$ , by Property 3.1, the estimate  $\gamma_j(\tau) = o(\tau^{\deg X_j})$  for all  $j > \dim H_1$  yields the  $hc$ -differentiability of  $\gamma$  at  $t_0$ .  $\square$

The method of proving Theorem 3.2.3 is applicable to a wider class of mappings and makes it possible to make additional conclusions about the nature of  $hc$ -differentiability.

**Corollary 3.2.4.** *Suppose that a curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  on a Carnot manifold is Lipschitz with respect to the Riemannian metric and horizontal, i.e.,  $\dot{\gamma}(s) \in H_{\gamma(s)}\mathbb{M}$  for almost every  $s \in [a, b]$ . Then the curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  is  $hc$ -differentiable almost everywhere<sup>3</sup>.*

*Proof.* Every Lipschitz curve with respect to the Riemannian metric is also absolutely continuous in the Riemannian sense. Thus all conditions of Theorem 3.2.3 hold.  $\square$

**Corollary 3.2.5.** *Suppose that we have a family of curves  $\gamma : [a, b] \times F \rightarrow \mathbb{M}$  on a Carnot manifold  $\mathbb{M}$  that is bounded and continuous in the totality of its variables, where  $F$  is a locally compact metric space. Suppose that,*

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<sup>3</sup>In papers [129, 130], a wrong Corollary 3.1 is formulates instead of this.

for each fixed  $u \in F$ , the curve  $\gamma(\cdot, u)$  is differentiable in the Riemannian sense at all points of  $[a, b]$  and horizontal, i.e.,  $\frac{d}{ds}\gamma(s, u) \in H_{\gamma(s, u)}\mathbb{M}$  for all  $s \in [a, b]$ . If the Riemannian derivative  $\frac{d}{ds}\gamma(s, u)$  is bounded and continuous in the totality of its variables  $s$  and  $u$  then its  $hc$ -derivative is also bounded and continuous on  $[a, b] \times F$ . Furthermore, the convergence  $\Delta_{\tau^{-1}}^{\gamma(s)}\gamma(s + \tau, u)$  to  $\dot{\gamma}(s, u) \in \mathcal{G}^{\gamma(s, u)}\mathbb{M}$  is locally uniform in the totality of  $s \in [a, b]$  and  $u \in F$ .

*Proof.* It suffices to prove in all items of the proof of Theorem 3.2.3 that the smallness of all quantities converging to zero is locally uniform on  $[a, b] \times F$  (see Proposition 2.8.6 for the estimate  $C_0 d_\infty(g, v) \leq d_c(g, v)$ )  $\square$

**Corollary 3.2.6.** *Suppose that a curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  on a Carnot manifold belongs to  $C^1$  and its Riemannian tangent vector  $\dot{\gamma}_i(t)$  is horizontal for all  $t \in [a, b]$ . Then the curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  is  $hc$ -differentiable at all  $t \in [a, b]$ . Furthermore, the convergence of  $\Delta_{\tau^{-1}}^{\gamma(s)}\gamma(s + \tau)$  to  $\dot{\gamma}(s) \in \mathcal{G}^{\gamma(s)}\mathbb{M}$  is uniform in  $s \in [a, b]$ .*

*Proof.* For any  $x, y \in [a, b]$ , the length  $L(\gamma|_{[x, y]})$  of the curve  $\gamma : [x, y] \rightarrow \mathbb{M}$  is defined; moreover,  $d_\infty(\gamma(y), \gamma(x)) \leq c_1 L(\gamma|_{[x, y]}) \leq c_1 C |y - x|$ , where  $C = \max_{t \in [a, b]} |\dot{\gamma}(t)|$ . Thus, the curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  meets the conditions of Theorem 3.2.3 at all points of  $[a, b]$  and, therefore, is uniformly  $hc$ -differentiable by Corollary 3.2.5. The last assertion of this corollary follows  $\square$

**Lemma 3.2.7.** *Every Lipschitz mapping  $\gamma : E \rightarrow \mathbb{M}$  is differentiable almost everywhere in the Riemannian sense, and  $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$  at the points of the Riemannian differentiability of  $\gamma$*

*Proof.* In the normal coordinates at a point  $g = \gamma(t)$ , we have

$$\gamma(t + \tau) = \exp\left(\sum_{j=1}^N \gamma_j(\tau) X_j\right)(g), \quad t + \tau \in E.$$

The Lipschitzity of the mapping  $\gamma : E \rightarrow \mathbb{M}$  and the properties of  $d_\infty$  imply the estimate  $\gamma_j(\tau) = O(\tau^{\deg X_j})$  for all  $j \geq 1, t + \tau \in E$ . Since  $\deg X_j \geq 2$  for  $j > \dim H_1$ , the derivative  $\dot{\gamma}_j(0)$  exists and is zero for all  $j > \dim H_1$ . Consequently, the Riemannian differentiability of  $\gamma$  at  $t$  is equivalent to the differentiability of the horizontal components  $\gamma_j, j = 1, \dots, n$ , of  $\gamma$  at 0.

Now, the Lipschitz mapping  $\gamma : E \rightarrow \mathbb{M}$  is also Lipschitz with respect to the Riemannian metric (see Proposition 2.8.12). Thus, by Rademacher's classical theorem, the Riemannian derivative  $\dot{\gamma}(t) \in T_{\gamma(t)}\mathbb{M}$  exists for almost every  $t \in [a, b]$ . The above implies that, at every such point,  $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$   $\square$



Since a Lipschitz mapping  $\gamma : [a, b] \rightarrow \mathbb{M}$  is absolutely continuous in the Riemannian sense (see the comparison of the metrics in Proposition 2.8.12), from Lemma 3.2.7 and Theorem 3.2.3 we infer

**Corollary 3.2.8.** *Every Lipschitz mapping  $\gamma : [a, b] \rightarrow \mathbb{M}$  is  $hc$ -differentiable almost everywhere on  $[a, b]$ : if  $t \in [a, b]$  is a Lebesgue point of the derivatives of its horizontal components then this point is its  $hc$ -differentiability point.*

### 3.2.3 $hc$ -Differentiability of scalar Lipschitz mappings

In this subsection, we establish the  $hc$ -differentiability of the Lipschitz mappings  $\gamma : E \rightarrow \mathbb{M}$  where  $E \subset \mathbb{R}$  is an arbitrary set.

Recall that  $x \in A$ , where  $A \subset \mathbb{R}$  is a measurable set, is the density point of  $A$  if

$$|A \cap (\alpha, \beta)|_1 = \beta - \alpha + o(\beta - \alpha) \quad \text{for } \beta - \alpha \rightarrow 0, x \in (\alpha, \beta)$$

(here  $|\cdot|_1$  stands for the one-dimensional Lebesgue measure). It is known that almost all points of a measurable set  $A$  are its density points (for example, see [41]).

It is explicitly seen from the above proof of Lemma 3.2.7 that the question of  $hc$ -differentiability for a Lipschitz mapping depends on the differentiability of its horizontal components. If a Lipschitz mapping  $\gamma : E \rightarrow \mathbb{M}$  (we may assume that  $E \subset \mathbb{R}$  is closed) is written in the normal coordinates:

$$\gamma(t + \tau) = \exp\left(\sum_{j=1}^N \gamma_j(\tau) X_j\right)(\gamma(t)), \quad t \in E \text{ is a fixed number, } t + \tau \in E,$$

then, by Lemma 3.2.7, its components  $\gamma_j(\tau)$ ,  $j = 1, \dots, N$ , are differentiable almost everywhere on  $E$ . It is known that almost all density points of  $E$  are Lebesgue points of the derivative of the horizontal components, i.e., for intervals  $(\alpha, \beta) \ni \tau$ ,  $t + \tau \in E$ , we infer

$$\int_{\{\sigma \in (\alpha, \beta) \mid t + \sigma \in E\}} |\dot{\gamma}_j(\sigma) - \dot{\gamma}_j(\tau)| d\sigma = o(\beta - \alpha) \quad \text{for } \beta - \alpha \rightarrow 0 \quad (3.2.11)$$

for all  $j = 1, \dots, \dim H_1$ . Note that property (3.2.11) does not depend on the choice of the coordinate system in a neighborhood of the point  $g = \gamma(t)$ .

**Theorem 3.2.9** ([129]). *Every Lipschitz mapping  $\gamma : E \rightarrow \mathbb{M}$ ,  $E \subset \mathbb{R}$  is closed, is  $hc$ -differentiable almost everywhere on  $E$ : the mapping  $\gamma : E \rightarrow \mathbb{M}$  is  $hc$ -differentiable at every point  $t \in E$  such that*

1.  $t$  is the density point of  $E$ ;

2. there exist derivatives  $\dot{\gamma}_j(0)$ ,  $j = 1, \dots, \dim H_1$ , of the horizontal components of  $\gamma$ , where  $\gamma(t + \tau) = \exp\left(\sum_{j=1}^N \gamma_j(\tau) X_j\right)(\gamma(t))$ ,  $t + \tau \in E$ ;
3. condition (3.2.11) is fulfilled at the point  $\tau = 0$ .

The hc-derivative  $\dot{\gamma}(t)$  equals

$$\exp\left(\sum_{j=1}^{\dim H_1} \dot{\gamma}_j(0) \widehat{X}_j^{\gamma(t)}\right)(\gamma(t)) = \exp\left(\sum_{j=1}^{\dim H_1} \dot{\gamma}_j(0) X_j\right)(\gamma(t)).$$

*Proof.* 1<sup>ST</sup> STEP. Suppose that  $t \in E$  is a point at which conditions 1–3 of the theorem hold and  $g = \gamma(t)$ . Since the result is local, we may also assume that  $E$  is included in an interval  $[a, b] \subset \mathbb{R}$ ,  $t \in [a, b]$ ,  $a, b \in E$ , whose image is included in  $\mathcal{G}^g \mathbb{M}$  (we may assume by diminishing the interval  $[a, b]$  if necessary that  $\gamma([a, b] \cap E) \subset \mathcal{G}^{\gamma(\eta)} \mathbb{M}$  for every  $\eta \in [a, b] \cap E$ ).

The open bounded set  $Z = (a, b) \setminus E$  is representable as the union of an at most countable collection of disjoint intervals:  $Z = \bigcup_j (\alpha_j, \beta_j)$ , where, for convenience of the subsequent estimates, we put  $\alpha_j < \beta_j$  if  $t \leq \alpha_j$  and  $\beta_j < \alpha_j$  if  $\alpha_j < t$ . It is known (for example, see [46]), that, in  $\mathcal{G}^{\gamma(\alpha_j)} \mathbb{M}$ , there exists a horizontal curve  $\tilde{\sigma}_j : [0, b_j] \rightarrow \mathcal{G}^{\gamma(\alpha_j)} \mathbb{M}$  joining the points  $\tilde{\sigma}_j(0) = \gamma(\alpha_j)$  and  $\tilde{\sigma}_j(b_j) = \gamma(\beta_j)$  and parameterized by the arc length; moreover,  $b_j = d_c^{\gamma(\alpha_j)}(\gamma(\alpha_j), \gamma(\beta_j)) \leq cd_\infty^{\gamma(\alpha_j)}(\gamma(\alpha_j), \gamma(\beta_j)) = cd_\infty(\gamma(\alpha_j), \gamma(\beta_j)) \leq cL|\beta_j - \alpha_j|$ , where  $c$  is independent of  $j$  (see the relation between the metrics in Subsection 2.8). Consequently, the mapping  $\sigma_j : [\alpha_j, \beta_j] \rightarrow \mathbb{M}$  defined by the rule

$$[\alpha_j, \beta_j] \ni \eta \mapsto \sigma_j(\eta) = \tilde{\sigma}_j\left(\frac{b_j}{|\beta_j - \alpha_j|}|\eta - \alpha_j|\right) \in \mathcal{G}^{\gamma(\alpha_j)} \mathbb{M}$$

is Lipschitz in the metric  $d_c^{\gamma(\alpha_j)}$  with the Lipschitz constant  $cL$  for all  $j \in \mathbb{N}$ . Define now the extension  $f : [a, b] \rightarrow \mathbb{M}$  as follows:

$$f(\eta) = \begin{cases} \gamma(\eta), & \text{if } \eta \in E, \\ \sigma_j(\eta), & \text{if } \eta \in (\alpha_j, \beta_j). \end{cases}$$

2<sup>ND</sup> STEP. The mapping  $f : [a, b] \rightarrow \mathbb{M}$  has the following properties:

- (1)  $f : [\alpha, \beta] \rightarrow \mathbb{M}$  is a Lipschitz mapping with respect to the Riemannian metric;
- (2) the Riemannian derivative of  $f$  exists for almost every  $\eta \in [a, b]$  and is bounded;
- (3) the vector  $\dot{f}(\eta)$  belongs to the horizontal space  $H_{\gamma(\eta)} \mathbb{M}$  for almost every  $\eta \in E$ ;

(4) the mapping  $f : [a, b] \rightarrow \mathbb{M}$  has a Riemannian derivative at  $t$  equal to  $\dot{\gamma}(t)$ ;

if  $f(t + \tau) = \exp \left( \sum_{j=1}^N f_j(\tau) X_j \right)(g)$ ,  $t + \tau \in [a, b]$ , then

(5)  $f_j(\tau) = O(\tau^{\deg X_j})$  as  $\tau \rightarrow 0$  for all  $j \geq 1$ ;

(6) 0 is a Lebesgue point for the derivatives  $\dot{f}_j(\tau)$ ,  $j = 1, \dots, \dim H_1$ .

Indeed, if  $t \leq \alpha_j < \eta_1 < \beta_j < \alpha_k < \eta_2 < \beta_k \leq b$  then, taking the relations between the metrics into account, we obtain the estimates  $\rho(f(\eta_1), f(\eta_2)) \leq \rho(f(\eta_1), \gamma(\beta_j)) + \rho(\gamma(\beta_j), \gamma(\alpha_k)) + \rho(\gamma(\alpha_k), f(\eta_2)) \leq C((\beta_j - \eta_1) + (\alpha_k - \beta_j) + (\eta_2 - \alpha_k)) = C|\eta_2 - \eta_1|$ . The other cases of mutual disposition of  $\eta_1$  and  $\eta_2$  with respect to  $t$  are considered similarly. Hence we obtain properties (1) and (2).

Next, if  $t \leq \alpha_j < t + \tau < \beta_j$  then  $d_\infty(f(t + \tau), f(t)) \leq C(d_\infty(f(t + \tau), \gamma(\alpha_j)) + d_\infty(\gamma(\alpha_j), \gamma(t))) \leq C_1(d_\infty^{(\alpha_j)}(f(t + \tau), \gamma(\alpha_j)) + (\alpha_j - t)) = C_2((t + \tau - \alpha_j) + (\alpha_j - t)) = C_2\tau$  by the triangle inequality, the construction of  $f$ , and the relations between the metrics. From this we obtain property (5) and, hence, the differentiability of all components  $f_j$  at 0,  $j > \dim H_1$ :  $\dot{f}_j(0) = 0$ .

Since the derivatives of Lipschitz functions are bounded and  $t$  is the density point of  $E$ , for intervals  $(r, s) \ni 0$  we have

$$\begin{aligned} \int_{(r,s)} |\dot{f}_j(\sigma) - \dot{\gamma}_j(0)| d\sigma &= \int_{\{\sigma \in (r,s) \mid t+\sigma \in E \cap [a,b]\}} |\dot{\gamma}_j(\sigma) - \dot{\gamma}_j(0)| d\sigma \\ &+ \int_{\{\sigma \in (r,s) \mid t+\sigma \notin E \cap [a,b]\}} |\dot{f}_j(\sigma) - \dot{\gamma}_j(0)| d\sigma = o(|s - r|) \quad (3.2.12) \end{aligned}$$

as  $s - r \rightarrow 0$  for all  $j = 1, \dots, \dim H_1$ . Hence,  $\int_0^\tau (\dot{f}_j(\sigma) - \dot{\gamma}_j(0)) d\sigma = f_j(\tau) - \dot{\gamma}_j(0)\tau = o(\tau)$  and  $\frac{df_j}{d\tau}(0) = \dot{\gamma}_j(0)$  for all  $j = 1, \dots, \dim H_1$ . Thus, we have proved properties (4) and (6).

Note that the preceding arguments are independent of the coordinate system. They are based on the following principle: if  $\eta$  is the density point for  $E$ , the mapping  $f|_E$  has a Riemannian derivative at  $\eta \in E$ , and  $\eta \in E$  is a Lebesgue point for the horizontal coordinate functions of  $f|_E$  then, with regard to Lemma 3.2.7 and what has been proved above,  $f$  has a Riemannian derivative at  $\eta$ ; moreover, the Riemannian tangent vector belongs to the horizontal space  $H_{\gamma(\eta)}\mathbb{M}$ . This proves property (3).

**3<sup>RD</sup> STEP.** Since the Riemannian derivative  $\dot{f}(\eta)$  of the mapping  $f : [a, b] \rightarrow \mathbb{M}$  belongs to the horizontal space  $H_{f(\eta)}\mathbb{M}$  only at almost every point  $\eta \in E$ , a direct application of Theorem 3.2.3 is impossible. However, granted

the fact that the complement  $[a, b] \setminus E$  has density zero at  $t$ , the method of its proof can be adapted also to this case. We now indicate the changes to the proof of Theorem 3.2.3 necessary for obtaining the  $hc$ -differentiability of  $f$  at the point  $t$  fixed above.

Introduce the notation

$$\Gamma(\tau) = \begin{cases} (\gamma_1(\tau), \dots, \gamma_N(\tau)), & \text{if } t + \tau \in E, \\ (f_1(\tau), \dots, f_N(\tau)), & \text{if } t + \tau \notin E. \end{cases}$$

It has been proved above that  $\dot{\Gamma}(0) = (\dot{\gamma}_1(0), \dots, \dot{\gamma}_{\dim H_1}(0), 0, \dots, 0)$ . Deduce (3.2.2) for the points  $\tau$  sufficiently close to 0 and such that  $t + \tau \in E$ . At the points  $t + \tau \in (\alpha_j, \beta_j)$ , we have

$$\dot{\Gamma}(\tau) = \sum_{j=1}^N \dot{f}_j(\tau) \frac{\partial}{\partial x_j} = \sum_{i=1}^{\dim H_1} a_i(\tau) \widehat{X}_i'^{f(\alpha_j)}(\Gamma(\tau)). \quad (3.2.13)$$

By Proposition 2.2.7, at the points  $t + \tau \in (\alpha_j, \beta_j)$  the relation  $f(\tau) \in B(g, O(\tau))$  implies that, in a neighborhood of 0, the vector fields  $\widehat{X}_i'^{f(\alpha_j)}$  are expressed via the vector fields  $\widehat{X}_k'$  (here we write  $\widehat{X}_k'$  instead of  $\widehat{X}_k'^g$ ) in the form

$$\widehat{X}_i'^{f(\alpha_j)}(\Gamma(\tau)) = \sum_{k=1}^N \gamma_{ik}(\tau) \widehat{X}_k'(\Gamma(\tau)), \text{ where } \gamma_{ik}(\tau) = \begin{cases} o(\tau^{\deg X_k - \deg X_i}), & \text{if} \\ \deg X_k > \deg X_i, \\ \delta_{ik} + o(1) & \text{otherwise} \end{cases}$$

as  $\tau \rightarrow 0$ . Really, by (2.2.6), we have  $\widehat{X}_i'^{f(\alpha_j)}(\Gamma(\tau)) = \sum_{l=1}^N \beta_{il}(\tau) \widetilde{X}_l(\Gamma(\tau))$  at points  $f(\tau) \in B(g, O(\tau))$ , where

$$\beta_{il}(\tau) = \begin{cases} o(\tau^{\deg X_l - \deg X_i}) & \text{if } \deg X_l > \deg X_i, \\ \delta_{il} + o(1) & \text{otherwise} \end{cases} \quad (3.2.14)$$

as  $\tau \rightarrow 0$ , and  $\widetilde{X}_l(\Gamma(\tau)) = \sum_{k=1}^N \alpha_{lk}(\tau) \widehat{X}_k'(\Gamma(\tau))$  where

$$\alpha_{lk}(\tau) = \begin{cases} o(\tau^{\deg X_k - \deg X_l}) & \text{if } \deg X_k > \deg X_l, \\ \delta_{lk} + o(1) & \text{otherwise} \end{cases} \quad (3.2.15)$$

as  $\tau \rightarrow 0$ . It follows  $\widehat{X}_i'^{f(\alpha_j)}(\Gamma(\tau)) = \sum_{k=1}^N \sum_{l=1}^N \beta_{il}(\tau) \alpha_{lk}(\tau) \widehat{X}_k'(\Gamma(\tau))$ . Now taking into account (3.2.14) and (3.2.15), and representing the last double sum as

$\sum_{k \leq i} \sum_{l=1}^N + \sum_{k > i} \left( \sum_{l \leq i} + \sum_{i < l \leq k} + \sum_{k < l} \right)$  we obtain the desired behavior of coefficients  $\gamma_{ik}(\tau)$  as  $\tau \rightarrow 0$ .

Consequently, we have just qualitative situation similar to those on the 3<sup>RD</sup> STEP of the proof of Theorem 3.2.3. Thus the theorem follows.  $\square$

### 3.2.4 *hc*-Differentiability of rectifiable curves

In this section, we in particular prove that, in a Carnot manifold, rectifiable curves are *hc*-differentiable almost everywhere. We obtain this result as a corollary to the more general assertion about the *hc*-differentiability of a mapping  $f : E \rightarrow \mathbb{M}$  from a measurable set  $E \subset \mathbb{R}$  that satisfies the condition

$$\overline{\lim}_{y \rightarrow x, y \in E} \frac{d_\infty(f(y), f(x))}{|y - x|} < \infty \quad (3.2.16)$$

for almost all  $x \in E$ .

**Theorem 3.2.10.** *Every mapping  $f : E \rightarrow \mathbb{M}$  satisfying (3.2.16) is *hc*-differentiable almost everywhere in  $E$ .*

*Proof.* Since the result is local, we may assume that  $E$  is bounded. Since, in view of (3.2.16), the “upper derivative” is finite almost everywhere, it follows that every point  $x \in E \setminus \Sigma$ , where  $\Sigma \subset E$  is some set of measure zero, belongs at least to one of the sets

$$A_k = \left\{ x \in E : \frac{d_\infty(f(x), f(y))}{|x - y|} \leq k \text{ for all } y \in E \cap (x - k^{-1}, x + k^{-1}) \right\}, \quad k \in \mathbb{N}. \quad (3.2.17)$$

Note that the sequence of sets  $A_k$  is monotone:  $A_k \subset A_{k+1}$ ,  $k \in \mathbb{N}$ . Suppose that the measure of  $A_k$  is nonzero for some  $k \in \mathbb{N}$ . Up to a set of measure zero, represent  $A_k$  as the union of a disjoint family of sets  $A_{k,1}, A_{k,2}, \dots$  of nonzero measure whose diameters are at most  $1/k$ :

$$A_k = Z_k \cup A_{k,1} \cup A_{k,2} \cup \dots, \quad |Z_k| = 0.$$

Then the restriction  $f_{k,j} = f|_{A_{k,j}}$  meets a Lipschitz condition for all  $j$ ; therefore, it is extendable by continuity to a Lipschitz mapping  $\tilde{f}_{k,j} : \overline{A_{k,j}} \rightarrow \mathbb{M}$ .

Verify that if  $(E \setminus \Sigma) \cap (\overline{A_{k,j}} \setminus A_{k,j}) \neq \emptyset$  then  $\tilde{f}_{k,j} : (E \setminus \Sigma) \cap \overline{A_{k,j}} \rightarrow \mathbb{M}$  coincides with  $f : (E \setminus \Sigma) \cap \overline{A_{k,j}} \rightarrow \mathbb{M}$ . In other words, if  $x \in (E \setminus \Sigma) \cap (\overline{A_{k,j}} \setminus A_{k,j})$  then the extension of  $f : A_{k,j} \rightarrow \mathbb{M}$  by continuity to the point  $x$  equals  $f(x)$ . Indeed, the chosen point  $x$  belongs  $E \setminus \Sigma$  and, therefore,  $x \in A_l$  for some  $l > k$ . Then the inequality of (3.2.17) holds for  $y \in E \cap (x - l^{-1}, x + l^{-1})$

with  $l$  instead of  $k$ . Since  $A_l \cap (x - l^{-1}, x + l^{-1}) \supset A_{k,j} \cap (x - k^{-1}, x + k^{-1})$ , we have

$$f(x) = \lim_{y \rightarrow x, y \in A_l} f(y) = \lim_{y \rightarrow x, y \in A_{k,j}} f(y) = \tilde{f}_{k,j}(x).$$

By Theorem 3.2.9, the mapping  $\tilde{f}_{k,j} : \overline{A}_{k,j} \rightarrow \mathbb{M}$  is  $hc$ -differentiable almost everywhere in  $\overline{A}_{k,j}$ . We are left with checking the  $hc$ -differentiability of the mapping  $f : E \rightarrow \mathbb{M}$  at the points of  $hc$ -differentiability of the mapping  $\tilde{f}_{k,j} : \overline{A}_{k,j} \rightarrow \mathbb{M}$  having density one with respect to  $\overline{A}_{k,j}$ .

For brevity, denote the set  $A_{k,j}$  by  $A$  and denote the mapping  $f_{k,j}$  by  $f$ . Extend the Lipschitz mapping  $f : A \rightarrow \mathbb{M}$  by continuity to a Lipschitz mapping  $\tilde{f} : \overline{A} \rightarrow \mathbb{M}$ .

Suppose now that a point  $a \in A$  is a point of  $hc$ -differentiability for  $\tilde{f}$  and the point density of  $\overline{A}$ . Recall that, by the definition of  $A$ , the inequality  $d_\infty(f(y), f(z)) \leq k|y - z|$  holds for all  $y \in A$  and all  $z \in (y - k^{-1}, y + k^{-1}) \cap E$ . Note that this inequality is extendable to  $\overline{A}$  by continuity. Consequently, the inequality

$$d_\infty(\tilde{f}(y), f(z)) \leq k|y - z|$$

holds for all  $y \in \overline{A}$  and all  $z \in (y - k^{-1}, y + k^{-1}) \cap E$ .

If  $z \in E$  belongs to the neighborhood  $(a - k^{-1}, a + k^{-1})$  of  $a$  then, by the well-known property of a density point (see, for example, [121]), there exists a point  $y \in \overline{A}$  such that  $|y - z| = o(|z - a|)$  as  $z \rightarrow a$ . Let  $X$  be the horizontal vector field of the definition of  $hc$ -differentiability for the restriction  $\tilde{f} : \overline{A} \rightarrow \mathbb{M}$  at a point  $a$ . Then, in a sufficiently small neighborhood of  $a$ , from what was said above we have

$$\begin{aligned} d_\infty(f(z), \exp((z-a)X)(f(a))) &\leq c^2(d_\infty(f(z), \tilde{f}(y)) + d_\infty(\tilde{f}(y), \exp(y-a)X)(f(a))) \\ &\quad + d_\infty(\exp(y-a)X)(f(a)), \exp(z-a)X)(f(a)) \\ &\leq c^2(k|y - z| + o(|y - a|) + \|X\||y - z|) = o(|z - a|) \end{aligned}$$

as  $z \rightarrow a$ ,  $z \in E$ . Hence, the mapping  $f : E \rightarrow \mathbb{M}$  is  $hc$ -differentiable at  $a$ .

Suppose now that  $k_1 < k_2 < k_3 \dots$  is a sequence of naturals such that the measure of the complement  $B_{k_j} = A_{k_j} \setminus A_{k_{j-1}}$  is nonzero for every  $j \geq 2$ . Obviously, the above argument applies to each of the sets  $B_{k_j}$ ,  $j \geq 2$ , which proves the theorem.  $\square$

Now we can prove the  $hc$ -differentiability of rectifiable curves. Consider a curve (continuous mapping)  $\gamma : [a, b] \rightarrow \mathbb{M}$ . By a partition  $I_n = I_n([a, b])$  of the segment  $[a, b]$  we mean any finite sequence of points  $\{s_1, \dots, s_n\}$  with  $a = s_1 < \dots < s_n = b$ . To every partition  $I_n([a, b])$ , we assign a number

$M(I_n)$  by setting

$$M(I_n) = \sum_{i=1}^n d_\infty(\gamma(s_i), \gamma(s_{i+1})).$$

Put  $m_n = \max\{s_{i+1} - s_i \mid i = 1, \dots, n-1\}$ .

**Definition 3.2.11** ([20]). A curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  is called *rectifiable* if

$$L([a, b]) = \lim_{m_n \rightarrow 0} \sup_{I_n} M_n < \infty.$$

Making use of a standard argument, we may prove:

**Property 3.2.12.** Suppose that a sequence of curves  $\gamma_n : [a, b] \rightarrow \mathbb{M}$ ,  $n \in \mathbb{N}$ , converges pointwise to a curve  $\gamma : [a, b] \rightarrow \mathbb{M}$ :  $\gamma_n(s) \rightarrow \gamma(s)$  for every  $s \in [a, b]$ . Then the lengths  $L_n([a, b])$  of  $\gamma_n$  possess the semicontinuity property:

$$L([a, b]) \leq \liminf_{n \rightarrow \infty} L_n([a, b]).$$

**Proposition 3.2.13.** Every rectifiable curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  meets (3.2.16).

*Proof.* Consider the following set function  $\Phi$  defined on intervals included in  $[a, b]$ : the value  $\Phi(\alpha, \beta)$  at an interval  $(\alpha, \beta) \subset [a, b]$  is equal to  $L([\alpha, \beta])$ , the length of the curve  $\gamma : [\alpha, \beta] \rightarrow \mathbb{M}$ . The set function  $\Phi$  is quasiadditive: the inequality

$$\sum_i \Phi(\alpha_i, \beta_i) \leq \Phi(\alpha, \beta)$$

holds for every finite collection of pairwise disjoint intervals  $(\alpha_i, \beta_i)$  with  $(\alpha_i, \beta_i) \subset (\alpha, \beta)$ , where  $(\alpha, \beta) \subset [a, b]$  is some interval. It is known (see, for example, [136]), that  $\Phi$  has a finite derivative

$$\Phi'(x) = \lim_{\substack{(\alpha, \beta) \ni x, \\ \beta - \alpha \rightarrow 0}} \frac{\Phi(\alpha, \beta)}{\beta - \alpha} = \lim_{\substack{(\alpha, \beta) \ni x, \\ \beta - \alpha \rightarrow 0}} \frac{L([\alpha, \beta])}{\beta - \alpha}$$

almost everywhere in  $[a, b]$ . Hence,

$$\overline{\lim}_{y \rightarrow x} \frac{d_\infty(f(y), f(x))}{|y - x|} \leq \overline{\lim}_{\substack{(\alpha, \beta) \ni x, \\ \beta - \alpha \rightarrow 0}} \frac{d_\infty(f(\alpha), f(\beta))}{L([\alpha, \beta])} \cdot \lim_{\substack{(\alpha, \beta) \ni x, \\ \beta - \alpha \rightarrow 0}} \frac{L([\alpha, \beta])}{\beta - \alpha} \leq \Phi'(x) < \infty$$

for almost all  $x \in [a, b]$ . □

Theorem 3.2.10 and Proposition 3.2.13 imply:

**Proposition 3.2.14.** Every rectifiable curve  $\gamma : [a, b] \rightarrow \mathbb{M}$  is *hc-differentiable* almost everywhere.

**Remark 3.2.15.** If the Carnot manifold is a Carnot group our definition of the  $hc$ -differentiability of curves coincides with the  $\mathcal{P}$ -differentiability of curves given by P. Pansu in [115]. He proved also [115, Proposition 4.1] the  $\mathcal{P}$ -differentiability almost everywhere of rectifiable curves on Carnot groups using a different method.

### 3.3 $hc$ -Differentiability of Smooth Mappings of Carnot Manifolds

#### 3.3.1 Continuity of horizontal derivatives and $hc$ -differentiability of mappings

In this subsection, we generalize the classical property that the continuity of the partial derivatives of a function defined on a Euclidean space guarantees its differentiability.

In what follows, we repeatedly use the following correspondence: to arbitrary element  $a = \exp\left(\sum_{i=1}^N a_i \widehat{X}_i^g\right)(g) \in \mathcal{G}^g$  and point  $w \in \mathcal{G}^g$ , assign the element

$$\Delta_\varepsilon^w a = \exp\left(\sum_{j=1}^N a_j \varepsilon^{\deg X_j} X_j\right)(w) \quad (3.3.1)$$

for those  $\varepsilon$  for which the right-hand side of (3.3.1) exists. Note that, by Property 2.2.3, we have  $\Delta_\varepsilon^g a = \delta_\varepsilon^g a$  for all  $a \in \mathcal{G}^g$ .

**Theorem 3.3.1.** *Suppose that  $f : \mathbb{M} \rightarrow \mathbb{N}$  is a Lipschitz mapping of Carnot manifolds such that, at each point  $g \in \mathbb{M}$ , there exist horizontal derivatives  $X_i f(g) \in H_{f(g)}\mathbb{N}$  continuous on  $\mathbb{M}$ ,  $i = 1, \dots, \dim H_1$ . Then  $f$  is  $hc$ -differentiable at every point of  $\mathbb{M}$ . The Lie algebra homomorphism corresponding to the  $hc$ -differential is uniquely defined by the mapping*

$$H_g \mathbb{M} \ni X_i(g) \mapsto X_i f(g) = \frac{d}{dt} f(\exp t X_i(g))|_{t=0} = \sum_{j=1}^{\dim \tilde{H}_1} b_{ij} Y_j(f(g)) \in H_{f(g)} \mathbb{N}$$

of the basis horizontal vectors  $X_i(g)$ ,  $i = 1, \dots, \dim H_1$ , to horizontal vectors in  $H_{f(g)}\mathbb{N}$ :

$$H\mathcal{G}^g \mathbb{M} \ni \widehat{X}_i^g \mapsto \sum_{j=1}^{\dim \tilde{H}_1} b_{ij} \widehat{Y}_j^{f(g)} \in H\mathcal{G}^{f(g)} \mathbb{N}.$$



*Proof.* 1<sup>ST</sup> STEP. Fix a point  $g \in U$  and a compact neighborhood  $F \subset \mathcal{G}^g$  of the local Carnot group  $\mathcal{G}^g$ . For each horizontal vector field  $X_i$ , a family of curves  $\gamma : [-\varepsilon, \varepsilon] \times F \rightarrow \mathbb{N}$  is defined: for  $u \in F$ , put  $\gamma_i(s, u) = f(\exp(s\alpha_i X_i)(u))$ , where  $\alpha_i \in A$ ,  $A \subset \mathbb{R}$  is a bounded neighborhood of  $0 \in \mathbb{R}$ . This family of curves meets the conditions of Corollary 3.2.5 and, hence, the convergence

$$\Delta_{s^{-1}}^{f(u)} \gamma(s, u) \rightarrow \delta_{\alpha_i}^{f(u)} \exp([X_i f](u))(f(u)) \in \mathcal{G}^{f(u)} \quad (3.3.2)$$

is uniform on  $F \times A$  and the  $hc$ -derivative  $\delta_{\alpha_i}^{f(u)} \exp(X_i f(u))(u)$  is continuous with respect to  $(u, \alpha_i) \in F \times A$ . Denote by  $x_i$  the “horizontal basis element”  $\exp(X_i)(g) = \exp(\hat{X}_i^g)(g) \in \mathcal{G}^g$  and, for all  $1 \leq i \leq \dim H_1$ , denote by  $a_i$  the horizontal derivative  $\exp(X_i f(g))(f(g))$ .

It is known [46] that any element  $v \in F$  can be represented (nonuniquely) in the form

$$\delta_{\alpha_1}^g x_{j_1} \cdots \delta_{\alpha_S}^g x_{j_S}, \quad 1 \leq j_i \leq \dim H_1, \quad (3.3.3)$$

where  $S$  is independent of the choice of the point and the numbers  $\alpha_i$  are bounded by a common constant. Together with the mapping

$$[0, \varepsilon) \ni t \mapsto \hat{v}_i(t) = \delta_{\alpha_1}^g x_{j_1} \cdots \delta_{\alpha_i}^g x_{j_i}, \quad 1 \leq j_k \leq \dim H_1, \quad 1 \leq k \leq i \leq S,$$

consider the mapping (see (3.3.1))

$$\begin{aligned} [0, \varepsilon) \ni t \mapsto v_i(t) &= \Delta_{\alpha_i}^{v_{i-1}(t)} x_{j_i} = \exp(t\alpha_i X_{j_i})(v_{i-1}(t)), \quad 2 \leq i \leq S, \text{ where} \\ v_1(t) &= \Delta_{\alpha_1}^g x_{j_1} = \exp(t\alpha_1 X_{j_1})(g). \end{aligned}$$

By Theorem 2.7.1,  $d_\infty(v_i(t), \hat{v}_i(t)) = o(t)$  as  $t \rightarrow 0$  uniformly in  $g \in F$  and  $\alpha_i \in A$ ,  $i \leq S$ . Since the mapping  $f$  is Lipschitz on  $F$ , the limits  $\lim_{t \rightarrow 0} \Delta_{t^{-1}}^{f(g)} f(\hat{v}_S(t))$  and  $\lim_{t \rightarrow 0} \Delta_{t^{-1}}^{f(g)} f(v_S(t))$  exist simultaneously. Consequently, it suffices to prove the existence of the second limit.

2<sup>ND</sup> STEP. For proving this, by (3.3.2), we infer that

$$\begin{aligned} w_1(t) &= f(v_1(t)) = \exp\left(\sum_{k=1}^{\tilde{N}} z_k^1(t) Y_k\right)(f(g)) \\ &\text{has } hc\text{-derivative } \delta_{\alpha_1}^{f(g)} a_{j_1} \in \mathcal{G}^{f(g)} \quad \text{at } t = 0. \end{aligned}$$

Here  $Y_k$ ,  $k = 1, \dots, \tilde{N}$ , is a local basis on  $\mathbb{N}$  around the point  $f(g)$ . Assume that the mapping

$$\begin{aligned} t \mapsto w_i(t) &= f(v_i(t)) = \exp\left(\sum_{k=1}^{\tilde{N}} z_k^i(t) Y_k\right)(f(v_{i-1}(t))) \\ &\text{has } hc\text{-derivative } \delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_i}^{f(g)} a_{j_i} \in \mathcal{G}^{f(g)}, \quad \text{at } t = 0, \quad 2 \leq i < S. \end{aligned}$$

Our next goal is to show that  $hc$ -derivative of the mapping  $t \mapsto w_{i+1}(t) = f(v_{i+1}(t)) = \exp\left(\sum_{k=1}^{\tilde{N}} z_k^{i+1}(t) Y_k\right)(f(v_i(t)))$  equals  $\delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_i}^{f(g)} a_{j_i} \cdot \delta_{\alpha_{i+1}}^{f(g)} a_{j_{i+1}}$ . Together with the mapping  $w_{i+1}(t)$ , consider the mapping

$$t \mapsto \widehat{w}_{i+1}(t) = \exp\left(\sum_{k=1}^{\tilde{N}} z_k^{i+1}(t) \widehat{Y}_k^g\right)(f(v_i(t))).$$

By Theorem 2.7.1 we have  $d_c^{f(g)}(w_{i+1}(t), \widehat{w}_{i+1}(t)) = o(t)$  as  $t \rightarrow 0$ . Therefore, the relation  $d_c^{f(g)}(w_{i+1}(t), \delta_t^{f(g)}(\delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_{i+1}}^{f(g)} a_{j_{i+1}})) = o(t)$  as  $t \rightarrow 0$  holds if and only if  $d_c^{f(g)}(\widehat{w}_{i+1}(t), \delta_t^{f(g)}(\delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_{i+1}}^{f(g)} a_{j_{i+1}})) = o(t)$  as  $t \rightarrow 0$ . By Property 3.1.2, this is equivalent to the relation

$$d_c^{f(g)}(\delta_{t-1}^g \widehat{w}_{i+1}(t), \delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_{i+1}}^{f(g)} a_{j_{i+1}}) = o(1) \quad \text{as } i \rightarrow \infty.$$

Note that, by the continuity of the group operation in  $\mathcal{G}^g$ , we always have the convergence

$$\delta_{t-1}^g \widehat{w}_{i+1}(t) \rightarrow \delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_{i+1}}^{f(g)} a_{j_{i+1}} \quad \text{as } t \rightarrow 0.$$

Thus, by induction, the  $hc$ -derivative of the mapping  $[0, \varepsilon) \ni t \mapsto f(v_S(t))$  at 0 is equal to  $\delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_S}^{f(g)} a_{j_S}$ ; moreover, the convergence is uniform in  $v \in F$  and  $\alpha_i$ ,  $1 \leq i \leq S$ . Consequently, granted the equality  $v_S(t) = \delta_t^g v$ , we infer

$$d_c^{f(g)}(f(\delta_t^g v), L(\delta_t^g v)) = o(d_c^g(g, \delta_t^g v)) = o(t) \quad (3.3.4)$$

uniformly in  $v \in F$ , where  $L$  stands for the correspondence

$$\mathcal{G}^g \ni v = \delta_{\alpha_1}^g x_{j_1} \cdots \delta_{\alpha_S}^g x_{j_S} \mapsto \delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_S}^{f(g)} a_{j_S} \in \mathcal{G}^{f(g)}.$$

For finishing the proof, it remains to check that the correspondence  $L : \mathcal{G}^g \rightarrow \mathcal{G}^{f(g)}$  is a homomorphism of the local Carnot groups.

**3<sup>RD</sup> STEP.** Note that  $L(v)$  is the  $hc$ -derivative at 0 of the mapping  $t \mapsto f(\delta_t^g v)$  for a fixed  $v \in \mathcal{G}^g$  (see (3.3.4)), which is obviously independent of representation (3.3.3). Consequently,  $L : \mathcal{G}^g \rightarrow \mathcal{G}^{f(g)}$  is a mapping of the local groups. Clearly, this mapping is continuous.

Demonstrate that it is a group homomorphism. Consider a second element  $\bar{v} = \delta_{\beta_1}^g x_{j_1} \cdots \delta_{\beta_S}^g x_{j_S}$ ,  $1 \leq j_i \leq \dim H_1$ , such that

$$v\bar{v} = \delta_{\alpha_1}^g x_{j_1} \cdots \delta_{\alpha_S}^g x_{j_S} \cdot \delta_{\beta_1}^g x_{j_1} \cdots \delta_{\beta_S}^g x_{j_S} \in \mathcal{G}^g \quad \text{and} \quad L(v) \cdot L(\bar{v}) \in \mathcal{G}^{f(g)}. \quad (3.3.5)$$

By (3.3.4), the value  $L(v\bar{v})$  is independent of the representation of an element  $v\bar{v}$  as the product (3.3.5). Hence, applying the conclusions of the previous step to  $v\bar{v}$  and its representation (3.3.5), we see that

$$L(v\bar{v}) = \delta_{\alpha_1}^{f(g)} a_{j_1} \cdots \delta_{\alpha_S}^{f(g)} a_{j_S} \cdot \delta_{\beta_1}^{f(g)} a_{j_1} \cdots \delta_{\beta_S}^{f(g)} a_{j_S} = L(v) \cdot L(\bar{v}).$$

Thus, the mapping  $L : \mathcal{G}^g \rightarrow \mathcal{G}^{f(g)}$  is a continuous group homomorphism. By the well-known properties of the Lie group theory [137], the mapping  $L$  is a homomorphism of the local Lie groups.

Now, from (3.3.4) it can be deduced that  $L$  commutes with a dilation,  $L \circ \delta_t^g = \delta_t^{f(g)} \circ L$ ,  $t > 0$ . Furthermore, since  $X_i f(g) \in H_{f(g)} \mathbb{M}$ , the homomorphism  $L$  is the *hc-differential of the mapping  $f : \mathbb{M} \rightarrow \mathbb{N}$*  at  $g$ . The Lie algebra homomorphism corresponding to  $L$  is a mapping of horizontal subspaces.  $\square$

**Corollary 3.3.2** ([129]). *Assume that we have a basis  $\{X_i\}$ ,  $i = 1, \dots, N$ , on a Carnot manifold  $\mathbb{M}$  for which Assumption 2.1.4 or conditions of Remark 2.7.2 hold with some  $\alpha \in (0, 1]$ . Suppose that  $f : \mathbb{M} \rightarrow \mathbb{N}$  is a mapping of Carnot manifolds such that, at each point  $g \in \mathbb{M}$ , there exist horizontal derivatives  $X_i f(g) \in H_{f(g)} \mathbb{N}$  continuous on  $\mathbb{M}$ ,  $i = 1, \dots, \dim H_1$ . Then  $f$  is hc-differentiable at every point of  $\mathbb{M}$ . The Lie algebra homomorphism corresponding to the hc-differential is uniquely defined by the mapping*

$$H_g \mathbb{M} \ni X_i(g) \mapsto X_i f(g) = \frac{d}{dt} f(\exp t X_i(g))|_{t=0} = \sum_{j=1}^{\dim \tilde{H}_1} b_{ij} Y_j(f(g)) \in H_{f(g)} \mathbb{N}$$

of the basis horizontal vectors  $X_i(g)$ ,  $i = 1, \dots, \dim H_1$ , to horizontal vectors in  $H_{f(g)} \mathbb{N}$ :

$$H\mathcal{G}^g \mathbb{M} \ni \hat{X}_i^g \mapsto \sum_{j=1}^{\dim \tilde{H}_1} b_{ij} \hat{Y}_j^{f(g)} \in H\mathcal{G}^{f(g)} \mathbb{N}.$$

*Proof.* The hypothesis implies that  $f$  is a locally Lipschitz mapping:  $\tilde{d}_\infty(f(x), f(y)) \leq C d_\infty(x, y)$ ,  $x, y$  belong to some compact neighborhood of  $U$ . To verify this, it suffices to join points  $x, y \in U$  by the horizontal curve  $\gamma$  of Subsection 2.8 whose length is controlled by the *hc*-distance  $d_\infty(x, y)$  and observe that  $f \circ \gamma$  is a horizontal curve whose length is controlled by the length of the initial curve. From this, Corollary 2.8.6 and Remark 2.8.7 we infer  $\tilde{d}_\infty(f(x), f(y)) \leq C_1 L(f \circ \gamma) \leq C_2 L(\gamma) \leq C_3 d_\infty(x, y)$ .  $\square$

### 3.3.2 Functorial property of tangent cones

The definition of the tangent cone depends on the local basis. The question arises on the connection between two tangent cones found from two different bases. The last Theorem 3.3 implies:

**Corollary 3.3.3** ([128, 129]). *Suppose that we have two local bases  $\{X_i\}$  and  $\{Y_i\}$ ,  $i = 1, \dots, N$ , on a Carnot manifold for both of which Assumption 2.1.4 or conditions of Remark 2.7.2 hold with some  $\alpha \in (0, 1]$ , and that two collections  $X_1, \dots, X_{\dim H_1}$  and  $Y_1, \dots, Y_{\dim H_1}$  generate the same horizontal subbundle  $H_1$ . Then the tangent cone  $\mathcal{G}^g$  defined by the  $\{X_i\}$ 's is isomorphic to the local Carnot group  $\tilde{\mathcal{G}}^g$ , determined by the  $\{Y_i\}$ 's:  $(\tilde{\delta}_{t-1}^g \circ \delta_t^g)(v)$  converges to an isomorphism of local Carnot groups  $\mathcal{G}^g$  and  $\tilde{\mathcal{G}}^g$  as  $t \rightarrow 0$  uniformly in  $v \in \mathcal{G}^g$ . (Here  $\tilde{\delta}_t^g$  is the one-parameter dilation group defined by the vector fields  $\{Y_i\}$ .)*

*The isomorphism of the Lie algebras corresponding to the hc-differential is defined uniquely by giving the mapping*

$$H_g \mathbb{M} \ni X_i(g) \mapsto X_i(g) = \sum_{j=1}^{\dim H_1} b_{ij} Y_j(g) \in H_g \mathbb{M}$$

*of the basis vectors  $X_i(g)$ ,  $i = 1, \dots, \dim H_1$ , of the horizontal space  $H_g \mathbb{M}$  to horizontal vectors of the space  $H_g \mathbb{M}$ :*

$$H\mathcal{G}^g \mathbb{M} \ni \hat{X}_i^g \mapsto \sum_{j=1}^{\dim H_1} b_{ij} \hat{Y}_j^g \in H\tilde{\mathcal{G}}^g \mathbb{M}.$$

*Proof.* Denote by  $\mathbb{M}^X$  the Carnot manifold  $\mathbb{M}$  with local basis  $\{X_i\}$  and denote by  $\mathbb{M}^Y$  the Carnot manifold  $\mathbb{M}$  with local basis  $\{Y_i\}$ ,  $i = 1, \dots, N$ . Let also the symbol  $i : \mathbb{M}^X \rightarrow \mathbb{M}^Y$  stand for the identity mapping from  $\mathbb{M}$  into  $\mathbb{M}$ . Clearly,  $i$  meets the conditions of Corollary 3.3.2. Then  $i$  is hc-differentiable at  $g$  and, by Corollary 3.3.2, the “difference ratios”  $\tilde{\delta}_{t-1}^g(\delta_t^g(w))$  converge uniformly to a homomorphism  $Di(g) : \mathcal{G}^g \rightarrow \tilde{\mathcal{G}}^g$  as  $t \rightarrow 0$ . Applying the same argument to the inverse mapping  $i^{-1}$  and Theorem 3.1.3, we infer that  $Di(g)$  is an isomorphism of the local Carnot groups (of the local tangent cones at  $g$  with respect to different local bases).  $\square$

**Remark 3.3.4.** In [4, 15, 66, 100] above statement is proved by other methods under additional assumptions on the smoothness of the basis vector fields.

### 3.3.3 Rademacher Theorem

The aim of this part is to formulate Rademacher type theorems on the differentiability of Lipschitz mappings of Carnot manifolds. This theorem was proved in [129] by means of the theory expounded above. The way of proving this result is based on the methods of [125], where the  $\mathcal{P}$ -differentiability of Lipschitz mappings of Carnot groups defined on measurable sets was proved in details.

Let  $\mathbb{M}, \mathbb{N}$  be two Carnot manifolds and let  $E \subset \mathbb{M}$  be an arbitrary set. A mapping  $f : E \rightarrow \mathbb{N}$  is called a Lipschitz mapping if

$$\tilde{d}_\infty(f(x), f(y)) \leq C d_\infty(x, y), \quad x, y \in E,$$

for some constant  $C$  independent of  $x$  and  $y$ . The least constant in this relation is denoted by  $\text{Lip } f$ .

The following result extends the theorems on the  $\mathcal{P}$ -differentiability on Carnot groups [115, 125, 135] (see also [93]) to Carnot manifolds.

**Theorem 3.3.5** ([129]). *Let  $E$  be a set in  $\mathbb{M}$  and let  $f : E \rightarrow \mathbb{N}$  be a Lipschitz mapping from  $E$  into  $\mathbb{N}$ . Then  $f$  is hc-differentiable on  $E$ .*

*The homomorphism of the Lie algebras corresponding to the hc-differential is defined uniquely by the mapping*

$$H_g \mathbb{M} \ni X_i(g) \mapsto X_i f(g) = \frac{d}{dt} f(\exp t X_i(g))|_{t=0} = \sum_{j=1}^{\dim \tilde{H}_1} a_{ij} Y_j(f(g)) \in H_{f(g)} \mathbb{N}$$

*of the horizontal basis vectors  $X_i(g)$ ,  $i = 1, \dots, \dim H_1$ , to horizontal vectors of the space  $H_{f(g)} \mathbb{N}$ :*

$$H\mathcal{G}^g \mathbb{M} \ni \hat{X}_i^g \mapsto \sum_{j=1}^{\dim \tilde{H}_1} a_{ij} \hat{Y}_j^{f(g)} \in H\mathcal{G}^{f(g)} \mathbb{N}.$$

### 3.3.4 Stepanov Theorem

As a corollary to Theorem 3.3.5, we obtain a generalization of Stepanov's theorem:

**Theorem 3.3.6** ([129]). *Let  $E \subset \mathbb{M}$  be a set in  $\mathbb{M}$  and let  $f : E \rightarrow \mathbb{N}$  be a mapping such that*

$$\overline{\lim}_{x \rightarrow a, x \in E} \frac{\tilde{d}_\infty(f(a), f(x))}{d_\infty(a, x)} < \infty$$

for almost all  $a \in E$ . Then  $f$  is *hc-differentiable* almost everywhere on  $E$  and the *hc-differential* is unique.

The homomorphism of the Lie algebras corresponding to the *hc-differential* is defined uniquely by the mapping

$$H_g \mathbb{M} \ni X_i(g) \mapsto X_i f(g) = \frac{d}{dt} f(\exp t X_i(g))|_{t=0} = \sum_{j=1}^{\dim \tilde{H}_1} a_{ij} Y_j(f(g)) \in H_{f(g)} \mathbb{N}$$

of the basis horizontal vectors  $X_i(g)$ ,  $i = 1, \dots, \dim H_1$ , to horizontal vectors of the space  $H_{f(g)} \mathbb{N}$ :

$$H\mathcal{G}^g \mathbb{M} \ni \hat{X}_i^g \mapsto \sum_{j=1}^{\dim \tilde{H}_1} a_{ij} \hat{Y}_j^{f(g)} \in H\mathcal{G}^{f(g)} \mathbb{N}.$$

## 4 Application: The Coarea Formula

### 4.1 Notations

All the above results on geometry and differentiability are applied in proving the sub-Riemannian analog of the well-known coarea formula for some classes of contact mappings of Carnot — Carathéodory spaces.

**Notation 4.1.1.** Denote by  $N_i$  the topological dimensions of  $\mathbb{M}_i$  and denote by  $\nu_i$  the Hausdorff dimensions of  $\mathbb{M}_i$ ,  $i = 1, 2$ . Assume that

$$T\mathbb{M}_1 = \bigoplus_{j=1}^{M_1} (H_j/H_{j-1}), \quad H_0 = \{0\}, \quad \text{and} \quad T\mathbb{M}_2 = \bigoplus_{j=1}^{M_2} (\tilde{H}_j/\tilde{H}_{j-1}), \quad \tilde{H}_0 = \{0\},$$

where  $H_1 \subset T\mathbb{M}_1$  and  $\tilde{H}_1 \subset T\mathbb{M}_2$  are *horizontal* subbundles. The subspace  $H_j \subset T\mathbb{M}_1$  ( $\tilde{H}_j \subset T\mathbb{M}_2$ ) is spanned by  $H_1$  ( $\tilde{H}_1$ ) and all commutators of order not exceeding  $j - 1$ ,  $j = 2, \dots, M_1$  ( $M_2$ ).

Denote the dimension of  $H_j/H_{j-1}$  ( $\tilde{H}_j/\tilde{H}_{j-1}$ ) by  $n_j$  ( $\tilde{n}_j$ ),  $j = 1, \dots, M_1$  ( $M_2$ ).

Here the number  $M_1$  ( $M_2$ ) are such that  $H_{M_1}/H_{M_1-1} \neq 0$  ( $\tilde{H}_{M_2}/\tilde{H}_{M_2-1} \neq 0$ ), and  $H_{M_1+1}/H_{M_1} = 0$  ( $\tilde{H}_{M_2+1}/\tilde{H}_{M_2} = 0$ ). The number  $M_1$  ( $M_2$ ) is called the *depth* of  $\mathbb{M}_1$  ( $\mathbb{M}_2$ ).

**Assumption 4.1.2.** Suppose that

1.  $N_1 \geq N_2$ ;
2.  $\dim H_i \geq \dim \tilde{H}_i$ ,  $i = 1, \dots, M_1$ ;

3. the basis vector fields  $X_1, \dots, X_{N_1}$  (in the preimage) are  $C^{1,\alpha}$ -smooth,  $\alpha > 0$ , and  $\tilde{X}_1, \dots, \tilde{X}_{N_2}$  (in the image) are  $C^{1,\varsigma}$ -smooth,  $\varsigma > 0$ , or conditions of Remark 2.7.2 hold for  $\alpha > 0$  in the preimage and  $\varsigma > 0$  in the image.

**Remark 4.1.3.** Note that, if there exists at least one point where the  $hc$ -differential  $\widehat{D}\varphi$  is non-degenerate, then the condition  $\dim H_1 \geq \dim \tilde{H}_1$  implies  $n_i \geq \tilde{n}_i$ ,  $i = 2, \dots, M_1$  (compare with the above assumption).

**Notation 4.1.4.** Denote by  $Z$  the set of points  $x \in \mathbb{M}_1$  such that  $\text{rank}(D\varphi(x)) < N_2$ .

## 4.2 Lay-out of the Proof

The key point in proving the non-holonomic coarea formula is to investigate the interrelation of "Riemannian" and Hausdorff measures on level sets (see below). The research on the comparison of "Riemannian" and Hausdorff dimensions of submanifolds of Carnot groups can be found in paper by Z. M. Balogh, J. T. Tyson and B. Warhurst [14]. See other results on sub-Riemannian geometric measure theory in works by L. Ambrosio, F. Serra Cassano and D. Vittone [12], L. Capogna, D. Danielli, S. D. Pauls and J. T. Tyson [28], D. Danielli, N. Garofalo and D.-M. Nhieu [34], B. Franchi, R. Serapioni and F. Serra Cassano [55, 56], B. Kirchheim and F. Serra Cassano [87], V. Magnani [94], S. D. Pauls [116] and many other.

The purpose of Section 4 is to explain the ideas of proof of the coarea formula for sufficiently smooth contact mappings  $\varphi : \mathbb{M}_1 \rightarrow \mathbb{M}_2$  of Carnot manifolds. Note that, all the obtained results are new even for the particular case of a mappings of Carnot groups.

**Remark 4.2.1.** For proving Theorems 4.2.6, 4.2.8, 4.2.9, and 4.2.13, the smoothness  $C^1$  (in Riemannian sense) for mappings  $\varphi : \mathbb{M}_1 \rightarrow \mathbb{M}_2$  is sufficient. For proving Theorem 4.2.12, the (Riemannian) smoothness  $C^{2,\varpi}$ ,  $\varpi > 0$ , of  $\varphi$  is sufficient.

As it is mentioned above, for the first time, a non-holonomic analogue of the coarea formula is proved in paper of P. Pansu [112]. The main idea of this work (which is used in many other ones) is to prove the coarea formula

via the Riemannian one:

$$\begin{aligned}
(1.0.3) &\implies \int_U \mathcal{J}_{N_2}^{Sb}(\varphi, x) d\mathcal{H}^{\nu_1}(x) \\
&= \int_{\mathbb{M}_2} d\mathcal{H}^{\nu_2}(z) \int_{\varphi^{-1}(z)} \frac{\mathcal{J}_{N_2}^{Sb}(\varphi, u)}{\mathcal{J}_{N_2}(\varphi, x)} d\mathcal{H}^{N_1-N_2}(u) \stackrel{?}{=} \int_{\mathbb{M}_2} d\mathcal{H}^{\nu_2}(z) \int_{\varphi^{-1}(z)} d\mathcal{H}^{\nu_1-\nu_2}(u)
\end{aligned} \tag{4.2.1}$$

Here  $N_1, N_2$  are topological dimensions, and  $\nu_1, \nu_2$  are Hausdorff dimensions of preimage and image, respectively; it is well-known that, in sub-Riemannian case, topological and Hausdorff dimensions differ. It easily follows from (4.2.1), that the key point in this problem is to investigate the interrelation of "Riemannian" and Hausdorff measures on Carnot manifolds themselves and on level sets of  $\varphi$ , and of Riemannian and sub-Riemannian coarea factors. It is well known that the question on interrelation of measures on Carnot manifolds is quite easy, while both the investigation of geometry of level sets and the calculation of sub-Riemannian coarea factor are non-trivial. The main problems are connected with peculiarities of a sub-Riemannian metric. In particular, the non-equivalence of Riemannian and sub-Riemannian metrics can be seen in the fact that "Riemannian" radius of a sub-Riemannian ball of a radius  $r$  varies from  $r$  to  $r^M$ ,  $M > 1$ , where the constant  $M$  depends on the Carnot manifold structure. Thus, a question arises immediately on how "sharp" the approximation of a level by its tangent plain is (since the "usual" order of tangency  $o(r)$  is obviously insufficient here: a level may "jump" from a ball earlier then it is expected). Also a question arises on existence of a such sub-Riemannian metric suitable for the description of the geometry of an intersection of a ball and a level set. But even if we answer these questions, one more question appears: what is the relation of the Hausdorff dimension of the image and measure of the intersection of a ball and a level set.

We have solved all the above problems. First of all, the points in which the differential is non-degenerate, are divided into two sets: regular and characteristic.

**Definition 4.2.2.** The set

$$\chi = \{x \in \mathbb{M}_1 \setminus Z : \text{rank } \widehat{D}\varphi(x) < N_2\}$$

is called the *characteristic* set. The points of  $\chi$  are called *characteristic*.

**Definition 4.2.3.** The set

$$\mathbb{D} = \{x \in \mathbb{M}_1 : \text{rank } \widehat{D}\varphi(x) = N_2\}$$

is called the *regular* set. If  $x \in \mathbb{D}$ , then we say that,  $x$  is a *regular* point.



We define a number  $\nu_0(x)$  depending on  $x \in \mathbb{M}$  that shows whether a point is regular or characteristic.

**Definition 4.2.4.** Consider the number  $\nu_0$  such that

$$\nu_0(x) = \min \left\{ \nu : \exists \{X_{i_1}, \dots, X_{i_{N_2}}\} \right. \\ \left. (\text{rank}([X_{i_j} \varphi](x))_{j=1}^{N_2} = N_2) \Rightarrow \left( \sum_{j=1}^{N_2} \deg X_{i_j} = \nu \right) \right\}.$$

It is clear that  $\nu_0|_{\chi} > \nu_2$  and  $\nu_0|_{\mathbb{D}} = \nu_2$ .

We also define such sub-Riemannian quasimetric  $d_2$ , that makes the calculation of measure of the intersection of a sub-Riemannian and a tangent plain to a level set possible:

**Definition 4.2.5.** Let  $\mathbb{M}$  be a Carnot manifold of topological dimension  $N$  and of depth  $M$ , and let  $x = \exp\left(\sum_{i=1}^{N_1} x_i X_i\right)(g)$ . Define the distance  $d_2(x, g)$  as follows:

$$d_2(x, g) = \max \left\{ \left( \sum_{j=1}^{n_1} |x_j|^2 \right)^{\frac{1}{2}}, \right. \\ \left. \left( \sum_{j=n_1+1}^{n_1+n_2} |x_j|^2 \right)^{\frac{1}{2 \cdot \deg X_{n_1+1}}}, \dots, \left( \sum_{j=N-n_M+1}^N |x_j|^2 \right)^{\frac{1}{2 \cdot \deg X_N}} \right\}.$$

The similar metric  $d_2^u$  is introduced in the local Carnot group  $\mathbb{G}_u \mathbb{M}$ .

The construction of  $d_2$  is based on the fact that a ball in this quasimetric  $\text{Box}_2$  asymptotically equals a Cartesian product of Euclidean balls:

$$\text{Box}_2(x, r) \approx B^{n_1}(x, r) \times B^{n_2}(x, r^2) \times \dots \times B^{n_M}(x, r^M), \quad M > 1,$$

where  $N, n_i, i = 1, \dots, M$ , are (topological) dimensions of balls. The latter fact makes the calculation of above-mentioned measure possible (while in the case when we replace balls by cubes, it is quite complicated since cubes have different shapes of sections).

Using properties of this quasimetric, we calculate the  $\mathcal{H}^{N_1-N_2}$ -measure of the intersection of a tangent plain to a level set and a sub-Riemannian ball in the quasimetric  $d_2$ .

**Theorem 4.2.6.** Fix  $x \in \varphi^{-1}(t)$ . Then, the  $\mathcal{H}^{N_1-N_2}$ -measure of the intersection  $T_0[(\varphi \circ \theta_x)^{-1}(t)] \cap \text{Box}_2(0, r)$  is equivalent to

$$C(1 + o(1))r^{\nu_1 - \nu_0(x)}$$

where  $C$  does not depend on  $r$ , and  $o(1) \rightarrow 0$  as  $r \rightarrow 0$ .

While investigating the approximation of a surface by its tangent plain, we introduce a "mixed" metric possessing some Riemannian and sub-Riemannian properties.

**Definition 4.2.7.** For  $v, w \in \text{Box}_2(0, r)$  put  $d_{2E}^0(v, w) = d_2^0(0, w - v)$ , where  $w - v$  denotes the Euclidean difference.

This definition implies that  $\text{Box}_2(0, r)$  coincides with a ball  $\text{Box}_{2E}(0, r)$  centered at 0 of radius  $r$  in the metric  $d_{2E}^0$ .

We prove that in regular points the tangent plain approximates the level set quite sharp with respect to this metric, and from here we deduce the possibility of calculation of the Riemannian measure of a level set and a sub-Riemannian ball intersection. Notable is the fact that this measure can be expressed via Hausdorff dimensions of the preimage and the image: it is equivalent to  $r^{\nu_1 - \nu_2}$  (see below):

**Theorem 4.2.8.** *Suppose that  $x \in \varphi^{-1}(t)$  is a regular point. Then:*

(I) *In the neighborhood of  $0 = \theta_x^{-1}(x)$ , there exists a mapping from  $T_0[(\varphi \circ \theta_x)^{-1}(t)] \cap \text{Box}_2(0, r(1 + o(1)))$  to  $\psi^{-1}(t) \cap \text{Box}_2(0, r)$ , such that both  $d_2$ - and  $\rho$ -distortions with respect to 0 equal  $1 + o(1)$ , where  $o(1)$  is uniform on  $\text{Box}_2(0, r)$ ;*

(II) *The  $\mathcal{H}^{N_1 - N_2}$ -measure of the intersection  $\varphi^{-1}(t) \cap \text{Box}_2(x, r)$  equals*

$$|g|_{\ker D\varphi(x)} \cdot \prod_{k=1}^{M_1} \omega_{n_k^1 - n_k^2} \cdot \frac{|D\varphi(x)|}{|\widehat{D}\varphi(x)|} r^{\nu_1 - \nu_2} (1 + o(1)),$$

where  $g$  is a Riemann tensor,  $\widehat{D}\varphi$  is the hc-differential of  $\varphi$ , and  $o(1) \rightarrow 0$  as  $r \rightarrow 0$ .

From these results and obtained properties, using a result of [136], we deduce the interrelation of two measures in regular points of a level sets.

**Theorem 4.2.9** (Measure Derivative on Level Sets). *Hausdorff measure  $\mathcal{H}^{\nu_1 - \nu_2}$  of the intersection  $\text{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x))$ , where  $x$  is a regular point, and  $\text{dist}(\text{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x)), \chi) > 0$ , asymptotically equals  $\omega_{\nu_1 - \nu_2} r^{\nu_1 - \nu_2}$ . The derivative  $D_{\mathcal{H}^{N_1 - N_2}} \mathcal{H}^{\nu_1 - \nu_2}(x)$  equals*

$$\frac{1}{|g|_{\ker D\varphi(x)}} \cdot \frac{\omega_{\nu_1 - \nu_2}}{\prod_{k=1}^{M_1} \omega_{n_k - \tilde{n}_k}} \cdot \frac{|\widehat{D}\varphi(x)|}{|D\varphi(x)|}.$$

Finally, we introduce the notion of the sub-Riemannian coarea factor via the values of the  $hc$ -differential of  $\varphi$ .

**Definition 4.2.10.** The *sub-Riemannian coarea factor* equals

$$\mathcal{J}_{N_2}^{SR}(\varphi, x) = |\widehat{D}\varphi(x)| \cdot \frac{\omega_{N_1}}{\omega_{\nu_1}} \frac{\omega_{\nu_2}}{\omega_{N_2}} \frac{\omega_{\nu_1 - \nu_2}}{\prod_{k=1}^{M_1} \omega_{n_k - \tilde{n}_k}}.$$

We consider and solve problems connected with the characteristic set. The case of characteristic points is a little more complicated since in characteristic points a surface may jump from a sub-Riemannian ball, consequently, we cannot estimate the measure of the intersection of the surface and the ball via the one of the tangent plain and the ball. Note also that in all the other works on sub-Riemannian coarea formula, the preimage has a group structure, which is essentially used in proving the fact that the Hausdorff measure of characteristic points on each level set equals zero (see also the paper [13] by Z. M. Balogh, dedicated to properties of the characteristic set). In the case of a mapping of two Carnot manifolds, there is no group structure neither in image, nor in preimage. Moreover, the approximation of Carnot manifold by its local Carnot group is insufficient for generalization of methods developed before. That is why we construct new "intrinsic" method of investigation of properties of the characteristic set. First of all, in all the characteristic points the  $hc$ -differential is degenerate. We solve this problem with the following assumption.

**Property 4.2.11.** Suppose that  $x \in \chi$ , and  $\text{rank } \widehat{D}\varphi(x) = N_2 - m$ . Let also  $\widehat{D}\varphi(x)$  equals zero on  $n_1 - \tilde{n}_1 + m_1$  horizontal (linearly independent) vectors,  $n_2 - \tilde{n}_2 + m_2$  (linearly independent) vectors from  $H_2/H_1$ ,  $n_k - \tilde{n}_k + m_k$  (linearly independent) vectors from  $H_k/H_{k-1}$ ,  $k = 3, \dots, M_2$ . Then, on the one hand, since  $\text{rank } \widehat{D}\varphi(x) = N_2 - m$ , we have  $\sum_{i=1}^{M_1} m_k = m$ . On the other hand,  $\text{rank } D\varphi(x) = N_2$ . Consequently, there exist  $m$  (linearly independent) vectors  $Y_1, \dots, Y_m$  of degrees  $l_1, \dots, l_{M_2}$  (which are minimal) from the kernel of the  $hc$ -differential  $\widehat{D}\varphi$ , such that  $D\varphi(x)(\text{span}\{H_{M_2}, Y_1, \dots, Y_m\}) = T_{\varphi(x)}\mathbb{M}_2$ .

In this subsection, we will assume that, among the vectors  $Y_1, \dots, Y_m$ ,  $m_1$  of them of the degree  $l_1$  have the horizontal image,  $m_2$  of them of the degree  $l_2 \geq l_1$  have image belonging to  $\tilde{H}_2$ , and  $m_k$  of them of the degree  $l_k$ ,  $l_k \geq l_{k-1}$ , have image belonging to  $\tilde{H}_k$ ,  $k = 3, \dots, M_2$ .

By another words, the "extra" vector fields on which the  $hc$ -differential of  $\varphi$  is degenerate in characteristic points, possess the following property: if in  $H_k/H_{k-1}(x)$  the quantity of such "extra" vectors equals  $m_k > 0$ , then

there exist  $m_k$  vectors from  $H_{l_k}/H_{l_k-1}(x)$  such that their images have the degree  $k$ , they are linearly independent with each other and with the images of  $H_{l_k-1}(x)$ ,  $l_k \geq l_{k-1}$ . We develop new "intrinsic" method of investigation of the properties of the characteristic set.

**Example.** The condition described in Assumption 4.2.11, is always valid for the following  $\mathbb{M}_1$  and  $\mathbb{M}_2$ :

1.  $\mathbb{M}_1$  is an arbitrary Carnot–Carathéodory space, and  $\mathbb{M}_2 = \mathbb{R}$ ;
2.  $M_1$  is an arbitrary Carnot–Carathéodory space of the topological dimension  $2m + 1$ ,  $\mathcal{G}^u \mathbb{M}_1 = \mathbb{H}^m$  for all  $u \in \mathbb{M}_1$ ,  $\mathbb{M}_2 = \mathbb{R}^k$ ,  $k \leq 2m$ ;
3.  $M_1 = M_2$ ,  $\dim H_1 \geq \dim \tilde{H}_1$ ,  $\dim(H_i/H_{i-1}) = \dim(\tilde{H}_i/\tilde{H}_{i-1})$ ,  $i = 2, \dots, M_1$ ;
4.  $M_1 = M_2 + 1$ ,  $\dim H_i = \dim \tilde{H}_i$ ,  $i = 1, \dots, M_2$ .

In particular, in Theorem 4.2.6 it is shown, that in the characteristic points  $\mathcal{H}^{N_1-N_2}$ -measure of the intersection of a sub-Riemannian ball and the tangent plain to the level set is equivalent to  $r$  to the power  $\nu_1 - \nu_0(x) < \nu_1 - \nu_2$ . Next, we show, that  $\mathcal{H}^{N_1-N_2}$ -measure of the intersection of the level set and the sub-Riemannian ball centered at a characteristic point is infinitesimally big in comparison with  $r^{\nu_1-\nu_2}$ , i. e., is equivalent to  $\frac{r^{\nu_1-\nu_2}}{o(1)}$  (but it is not necessarily equivalent to  $r^{\nu_1-\nu_0(x)}$ ). From here we deduce that, the intersection of the characteristic set with each level set has zero  $\mathcal{H}^{\nu_1-\nu_2}$ -measure.

**Theorem 4.2.12** (Size of the Characteristic Set). *The Hausdorff measure  $\mathcal{H}^{\nu_1-\nu_2}(\chi \cap \varphi^{-1}(t)) = 0$  for all  $t \in \mathbb{M}_2$ .*

We also show that the degenerate set of the differential does not influence both parts of the coarea formula.

**Theorem 4.2.13.** *For  $\mathcal{H}^{\nu_2}$ -almost all  $t \in \mathbb{M}_2$ , we have*

$$\mathcal{H}^{\nu_1-\nu_2}(\varphi^{-1}(t) \cap Z) = 0.$$

Finally, we deduce the sub-Riemannian coarea formula.

**Theorem 4.2.14.** *For any smooth contact mapping  $\varphi : \mathbb{M}_1 \rightarrow \mathbb{M}_2$  possessing Property 4.2.11, the coarea formula holds:*

$$\int_{\mathbb{M}_1} \mathcal{J}_{N_2}^{Sb}(\varphi, x) d\mathcal{H}^{\nu_1}(x) = \int_{\mathbb{M}_2} d\mathcal{H}^{\nu_2}(t) \int_{\varphi^{-1}(t)} d\mathcal{H}^{\nu_1-\nu_2}(u).$$

As an application, using the result of the paper by R. Monti and F. Serra Cassano [107, Theorem 4.2] for Lip-functions defined on a Carnot–Carathéodory space  $\mathbb{M}$  of the Hausdorff dimension  $\nu$ , we deduce that the De Giorgi perimeter coincides with  $\mathcal{H}^{\nu-1}$ -measure on almost every level of a smooth function  $\varphi : \mathbb{M}$ :

**Theorem 4.2.15.** *For  $C^{2,\alpha}$ -functions  $\varphi : \mathbb{M} \rightarrow \mathbb{R}$ ,  $\alpha > 0$ , where  $\dim_{\mathcal{H}} \mathbb{M} = \nu$ , the De Giorgi perimeter coincides with  $\mathcal{H}^{\nu-1}$ -measure on almost every level.*

## 5 Appendix

### 5.1 Proof of Lemma 2.1.13

*Proof.* It is well known, that the solution  $y(t, u)$  of the ODE (2.1.6) equals  $y(t, u) = \lim_{n \rightarrow \infty} y_n(t, u)$ , where

$$y_0(t, u) = \int_0^t f(y(0), u) d\tau, \quad \text{and} \quad y_n(t, u) = \int_0^t f(y_{n-1}(\tau, u), u) d\tau.$$

This convergence is uniform in  $u$ , if  $u$  belongs to some compact set.

From the definition of this sequence it follows, that  $y_n(t) \rightarrow y(t)$  as  $n \rightarrow \infty$  in  $C^1$ -norm.

**1.** We show, that every  $y_n(t, u) \in C^\alpha(u)$  for each  $t \in [0, 1]$ . We have

$$\begin{aligned}
\max_t |y_n(t, u_1) - y_n(t, u_2)| &\leq \int_0^1 |f(y_{n-1}(\tau, u_1), u_1) - f(y_{n-1}(\tau, u_2), u_2)| d\tau \\
&\leq \int_0^1 |f(y_{n-1}(\tau, u_1), u_1) - f(y_{n-1}(\tau, u_1), u_2)| d\tau \\
&\quad + \int_0^1 |f(y_{n-1}(\tau, u_1), u_2) - f(y_{n-1}(\tau, u_2), u_2)| d\tau \\
&\leq H(f)|u_1 - u_2|^\alpha + L \max_t |y_{n-1}(t, u_1) - y_{n-1}(t, u_2)| \\
&\leq H(f) \sum_{m=0}^{n-1} L^m |u_1 - u_2|^\alpha + L^n \max_t |y_0(t, u_1) - y_0(t, u_2)| \\
&\leq H(f) \sum_{m=0}^{\infty} L^m |u_1 - u_2|^\alpha,
\end{aligned}$$

where  $H(f)$  is a constant, such that  $|f(u_1) - f(u_2)| \leq H(f)|u_1 - u_2|^\alpha$ . Note that the constant  $H = H(f) \sum_{m=0}^{\infty} L^m < \infty$  since  $L < 1$ , and it does not depend on  $n \in \mathbb{N}$ .

Suppose that  $u$  belongs to some compact set  $U$ . Then

$$\begin{aligned}
|y(t, u_1) - y(t, u_2)| &\leq |y(t, u_1) - y_n(t, u_1)| + |y_n(t, u_1) - y_n(t, u_2)| + |y(t, u_2) - y_n(t, u_2)| \\
&\leq H|u_1 - u_2|^\alpha + 2\varepsilon
\end{aligned}$$

for every  $\varepsilon = \varepsilon(n) > 0$ . Since the convergence is uniform in  $u \in U$ , and  $\varepsilon(n) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $|y(t, u_1) - y(t, u_2)| \leq H|u_1 - u_2|^\alpha$ , and  $y \in C^\alpha(u)$  locally.

To show, that  $\frac{\partial y}{\partial v_i}(t, v, u) \in C^\alpha(u)$  locally,  $i = 1, \dots, N$ , we obtain our estimates in the simplest case of  $N = 1$ .

**2.** Note that the mappings  $\{y_n\}_{n \in \mathbb{N}}$  converge to  $y$  in  $C^1$ -norm, and this convergence is uniform, if  $u$  belongs to some compact set  $U$ .

Let  $u \in U$ ,  $v \in W(0) \subset \mathbb{R}^N$ . Then similarly to the case **1**, we see, that if

the Hölder constant of  $y'_n$  does not depend on  $n \in \mathbb{N}$ , then  $y' \in C^\alpha(u)$ .

$$\begin{aligned}
& \max_{t,v} \left| \frac{dy_n}{dv}(t, v, u_1) - \frac{dy_n}{dv}(t, v, u_2) \right| \\
& \leq \max_{t,v} \left| \frac{d}{dv} \int_0^t f(y_{n-1}(\tau, v, u_1), v, u_1) - f(y_{n-1}(\tau, v, u_2), v, u_2) d\tau \right| \\
& \leq \max_{t,v} \left| \frac{d}{dv} \int_0^t f(y_{n-1}(\tau, v, u_1), v, u_1) - f(y_{n-1}(\tau, v, u_2), v, u_1) d\tau \right| \\
& + \max_{t,v} \left| \frac{d}{dv} \int_0^t f(y_{n-1}(\tau, v, u_2), v, u_1) - f(y_{n-1}(\tau, v, u_2), v, u_2) d\tau \right|. \quad (5.1.1)
\end{aligned}$$

For the first summand we have

$$\begin{aligned}
& \max_{t,v} \left| \frac{d}{dv} \int_0^1 f(y_{n-1}(\tau, v, u_1), v, u_1) - f(y_{n-1}(\tau, v, u_2), v, u_1) d\tau \right| \\
& \leq \max_v \int_0^1 \left| \frac{d}{dv} (f(y_{n-1}(\tau, v, u_1), v, u_1) - f(y_{n-1}(\tau, v, u_2), v, u_1)) \right| d\tau \\
& \leq \max_v \int_0^1 \left| \frac{df}{dy} \frac{dy_{n-1}}{dv}(\tau, v, u_1) - \frac{df}{dy} \frac{dy_{n-1}}{dv}(\tau, v, u_2) \right| d\tau \\
& + \max_v \int_0^1 \left| \frac{\partial f}{\partial v}(y_{n-1}(\tau, v, u_1)) - \frac{\partial f}{\partial v}(y_{n-1}(\tau, v, u_2)) \right| d\tau. \quad (5.1.2)
\end{aligned}$$

Then, we get

$$\max_v \int_0^1 \left| \frac{\partial f}{\partial v}(y_{n-1}(\tau, v, u_1)) - \frac{\partial f}{\partial v}(y_{n-1}(\tau, v, u_2)) \right| d\tau \leq C(f)H(y)|u_1 - u_2|^\alpha,$$

since each  $y_m$  is Hölder. The first summand in (5.1.2) is evaluated in the

following way:

$$\begin{aligned}
& \max_v \int_0^1 \left| \frac{df}{dy} \frac{dy_{n-1}}{dv}(\tau, v, u_1) - \frac{df}{dy} \frac{dy_{n-1}}{dv}(\tau, v, u_2) \right| d\tau \\
& \leq \max_v \int_0^1 \left| \frac{df}{dy}(u_1) \frac{dy_{n-1}}{dv}(\tau, v, u_1) - \frac{df}{dy}(u_1) \frac{dy_{n-1}}{dv}(\tau, v, u_2) \right| d\tau \\
& \quad + \max_v \int_0^1 \left| \frac{df}{dy}(u_1) \frac{dy_{n-1}}{dv}(\tau, v, u_2) - \frac{df}{dy}(u_2) \frac{dy_{n-1}}{dv}(\tau, v, u_2) \right| d\tau \\
& \leq L \max_{t,v} \left| \frac{dy_{n-1}}{dv}(t, v, u_1) - \frac{dy_{n-1}}{dv}(t, v, u_2) \right| \\
& \quad + \max_{u,v} \int_0^1 \left| \frac{dy_{n-1}}{dv}(\tau, v, u) \right| d\tau \cdot H(Df) |u_1 - u_2|^\alpha. \quad (5.1.3)
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
\max_{u,v} \int_0^1 \left| \frac{dy_m}{dv}(\tau, v, u) \right| d\tau & \leq \max_{t,u,v} \left| \frac{dy_m}{dv}(t, v, u) \right| \\
& = \max_{t,u,v} \left[ L \left| \frac{dy_{m-1}}{dv} \right| + \left| \frac{\partial f}{\partial v} \right| \right] \leq \max_{t,u,v} \left| \frac{\partial f}{\partial v} \right| \left[ \sum_{k=0}^{\infty} L^k \right] < \infty.
\end{aligned}$$

Thus, in the first summand of (5.1.1) we have

$$L \max_{t,v} \left| \frac{dy_{n-1}}{dv}(t, v, u_1) - \frac{dy_{n-1}}{dv}(t, v, u_2) \right| + C |u_1 - u_2|^\alpha,$$

where  $0 < C < \infty$  does not depend on  $n \in \mathbb{N}$ . The second summand in (5.1.1) is

$$\begin{aligned}
& \max_{t,v} \left| \frac{d}{dv} \int_0^t f(y_{n-1}(\tau, v, u_2), v, u_1) - f(y_{n-1}(\tau, v, u_2), v, u_2) d\tau \right| \\
& \max_v \int_0^1 \left| \frac{\partial f}{\partial v}(y_{n-1}, v, u_1) - \frac{\partial f}{\partial v}(y_{n-1}, v, u_2) \right| d\tau \leq C(f) |u_1 - u_2|.
\end{aligned}$$



Thus,

$$\begin{aligned}
\max_{t,v} \left| \frac{dy_n}{dv}(t, v, u_1) - \frac{dy_n}{dv}(t, v, u_2) \right| \\
\leq L \max_{t,v} \left| \frac{dy_{n-1}}{dv}(t, v, u_1) - \frac{dy_{n-1}}{dv}(t, v, u_2) \right| + K|u_1 - u_2|^\alpha \\
\leq k \sum_{k=0}^{\infty} L^k |u_1 - u_2|^\alpha,
\end{aligned}$$

and  $\frac{dy_n}{dv} \in C^\alpha(u)$  locally. Hence,  $\frac{dy}{dv} \in C^\alpha(u)$  locally.  $\square$

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